AN ELEMENTARY PROOF OF 2-CATEGORICAL PASTING

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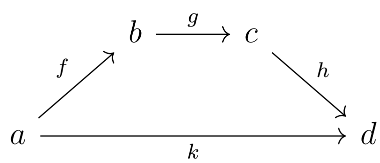
Abstract. This note provides an elementary proof that any pasting diagram in a 2-category has a uniquely defined 2-cell composite.

1. Introduction

The 2-categorical pasting theorem allows for a diagrammatic calculus in 2-category theory analogous to the use of commutative diagrams in 1-category theory.

In ordinary 1-category theory, diagrams allow for a visual representation of data. One of the most basic observations about a diagram is that it commutes. Making such a judgment involves considering the diagram, *e.g.*

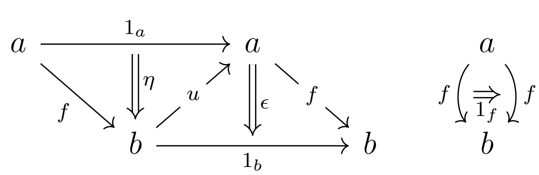


extracting strings of composable morphisms, *e.g.*

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

and considering their composite. Since composition in a category is associative, there is no need to fret about how the composite is obtained.

In a 2-category, the role of strings of composable morphisms is played by **pasting diagrams** and the basic judgment is whether two pasting diagrams have equal **composite**. For instance, the following are two pasting diagrams



(1)

The equality of their composites is part of the condition that f is left adjoint to u with unit η and counit ϵ.

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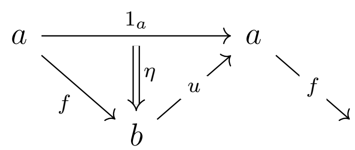
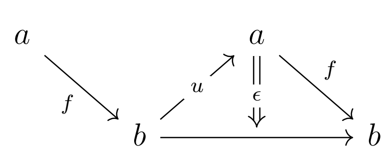
The composite of a pasting diagram is a 2-morphism. To extract it, one breaks the pasting diagram into simpler pieces, *e.g.*



and



finds 2-morphisms composites for these by whiskering, *e.g.* obtaining $f * \eta$ and $\epsilon * f$, and then composing these to obtain the overall composite 2-morphism, *e.g.* $(\epsilon * f) \circ (f * \eta)$.

There are two problems with this story.

(P1) One must actually define pasting diagram. The above discussion, at best, describes a pasting diagram as a labeled "picture that looks like this" which is not precise. Related to this is the problem of obtaining a precise definition of the composite of a pasting diagram. In the above, we refer to "simple pieces," but this must be made precise.

(P2) For a complex pasting diagram there may be many ways to obtain a decomposition into simple pieces. It must be shown that different decompositions lead to the same composite. The available associativity-like axioms for 2-categories do not immediately resolve this problem.

A solution to these problems first appeared in [Pow90]. A detailed textbook treatment in [JY21] follows similar ideas. To resolve (P1), Power defined a pasting diagram to be a directed graph, equipped with an explicit planar embedding, satisfying various properties, and equipped with certain labels. For instance, the bounded faces of the plane graph are labeled with 2-morphisms. Power used the planar structure and assumed properties to define the composite of such a pasting diagram and proved that it is unique, resolving (P2).

This treatment of pasting diagrams resolves the problems outlined above quite cleanly. A pasting diagram is defined to be - literally - a picture with specific well-defined properties in such a way that any picture of a pasting diagram is clearly a pasting diagram. The present author is interested in two features of this account.

(R1) The significant features of a pasting diagram, thought of as "a picture that looks like this," is a finite amount of combinatorial data. It should be a very simple object. However, a graph equipped with an explicit plane embedding is far from simple. It encodes a great deal of information which is not necessary for a pasting diagram to carry. In fact, a pasting diagram in the sense of Power is not really a plane graph. Rather, it should be thought of as an isotopy class of plane graphs. Still, an equivalence class of plane embeddings is not a simple object.

(R2) Working with pasting diagrams as plane graphs requires non-trivial topology. For instance, the Jordan Curve Theorem is needed to prove that the faces of a plane graph exist in general. A foundational result in 2-category theory relying on non-trivial topology is an odd state of affairs.

The above motivated the author to find a definition of pasting diagram with the following goals in mind

(G1) A pasting diagram must consist of a reasonably small amount of combinatorial data.

(G2) Pictures as in (1) must be unambiguously recognized as notation for pasting diagrams.

(G3) Pasting diagrams must have a unique 2-morphism composite.

(G4) Any pasting diagram in the sense of [JY21] must correspond to a pasting diagram in this new sense and have the same composite.

In this paper, a pasting diagram will be defined using computads, introduced in [Str76]. Any 2-category $\mathcal{C}$ has an underlying computad $U\mathcal{C}$. Any computad Γ generates a free category $F\Gamma$. These operations fit into an adjunction

$$2\mathbf{cat} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{cmptd} \quad (F \dashv U)$$

where $2\mathbf{cat}$ is the category of small 2-categories and **cmptd** is the category of computads. Certain nice computads equipped with data will be called **pasting schemes**. Given a 2-category $\mathcal{C}$ and a computad Γ, it is an easy combinatorial task to define a computad morphism $\Gamma \to U\mathcal{C}$ or equivalently a 2-functor $\Phi : F\Gamma \to \mathcal{C}$. When Γ is a pasting scheme, we will call such a 2-functor Φ a **pasting diagram of shape Γ in $\mathcal{C}$**. The extra data carried by the pasting scheme Γ specifies two distinguished morphisms $p, q \in F\Gamma$ such that $\mathrm{Hom}_{F\Gamma}(p, q) = \{\alpha_\Gamma\}$ is a singleton. The **composite** 2-morphism of the pasting diagram Φ is defined to be $\Phi(\alpha_\Gamma)$.

In Section 2, we recall the necessary facts about computads from [Str76]. In Section 3, we introduce the conditions on computads which define pasting schemes, meeting goals (G1) and (G2). We prove that certain sets of 2-morphisms in the free 2-category generated by a pasting scheme are singletons. This, along with the definition of composite, obtains goal (G3). In Section 4, we compare our pasting diagrams with those in [JY21], meeting goal (G4).

1.1. Remark. There are other combinatorial approaches to categorical pasting in the literature; see the torsion free complexes of [For22] and the other models discussed there. These have been used to address pasting theorems in the (∞, n)-categorical context, *c.f.*

[Cam23]. As far as the author is aware, none of these address the specific goal of this paper of providing an elementary and combinatorial proof of the 2-categorical pasting theorem. In [Joh89] - which shares our terminology for pasting schemes and diagrams - Observation 15 (following from Theorem 13) offers a pasting theorem for ω-categories. However, [For22] points out errors in Theorem 13 which throws doubt on the pasting theorem. Moreover, this whole framework is designed for the ω-categorical context and is more complicated and abstract than our treatment. In [Joh89], Johnson writes that he intends to produce a version of that paper purely in the 2-categorical context. To the author's knowledge, this did not occur.

2. Computads

In this section, we recall the necessary theory of computads from [Str76]. Throughout, graph means directed graph, possibly with multiple-edges and loops. In this spirit, path means directed path. Cycle means directed cycle. Etc.

2.1. Notation. *Paths are written in composition order. That is, the path*

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$$

is written (g, f)*. Concatenation of paths is written as* $*$*. Thus,* $(g, f) = g * f$*.*

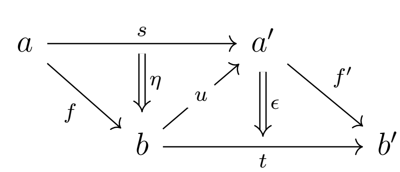
2.2. Definition. *A graph is* ***trim*** *when the source and target maps are jointly surjective; that is, each vertex must be the source or target of some edge.*

2.3. Definition. *A* ***computad*** Γ *consists of a graph* Γ_0 *and for each vertex* a, b *of* Γ_0 *a trim graph* $\Gamma(a, b)$ *whose vertices are paths from* a *to* b *in* Γ_0*. The edges in* $\Gamma(a, b)$ *are referred to as* ***2-edges****.*

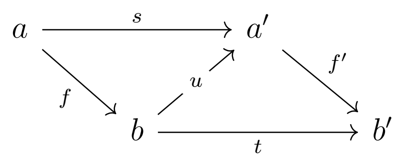
A ***morphism*** *of computads* $F : \Gamma \to \Gamma'$ *is a graph morphism* $F : \Gamma_0 \to \Gamma'_0$ *further equipped with graph morphisms* $F : \Gamma(a, b) \to \Gamma'(Fa, Fb)$ *for each pair of vertices* a, b *in* Γ_0*. We require that the action of* $F : \Gamma(a, b) \to \Gamma'(Fa, Fb)$ *on vertices is induced from the action of* $F : \Gamma_0 \to \Gamma'_0$ *on paths. The resulting category of computads and morphisms is denoted* **cmptd***.*

2.4. Notation. *We use* s *and* t *to denote the source and target operations on both the underlying graph of a computad and on the graphs whose vertices are paths. Thus, if* Γ *is a computad and* $e \in \Gamma_0$ *is an edge or path, then* $s(e)$ *is the source vertex of* e *in* Γ_0*. If* α *is a 2-edge in* $\Gamma(a, b)$ *then* $s(\alpha)$ *is its source, a path in* Γ_0 *from* a *to* b*.*

2.5. Example. Consider a picture of the form



which is a pasting scheme of the "picture like this" sort. This can be seen as a drawing of computad Γ whose underlying graph Γ_0 is



and whose only non-empty graphs of the form $\Gamma(x, y)$ are $\Gamma(a, a')$ and $\Gamma(b, b')$. These are the graphs

$$s \longrightarrow (u, f) \qquad\qquad (f', u) \longrightarrow t$$

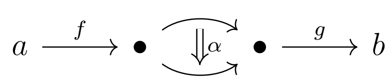
This sort of computad is how we will achieve goal (G2).

2.6. Example. Consider a 2-category $\mathcal{C}$. This generates an **underlying computad** $U\mathcal{C}$ as follows. The graph $(U\mathcal{C})_0$ is the graph underlying the 1-category underlying $\mathcal{C}$. Its vertices are the objects of $\mathcal{C}$ and the edges are the morphisms. Given paths $f = (f_n, f_{n-1}, ..., f_0)$ and $g = (g_m, g_{m-1}, ..., g_0)$ in $(U\mathcal{C})_0$, a 2-edge in $(U\mathcal{C})(f, g)$ is exactly a 2-morphism $f_n \circ \cdots \circ f_0 \Rightarrow g_m \circ \cdots \circ g_0$ in $\mathcal{C}$. Observe that U extends to a functor $U : 2\mathbf{cat} \to \mathbf{cmptd}$.

Next, we will define a free 2-category $F\Gamma$ generated by a computad Γ. This construction will extend to a functor $F : \mathbf{cmptd} \to 2\mathbf{cat}$ left adjoint to U.

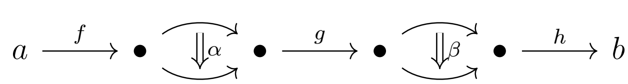
2.7. Construction. *Fix a computad Γ and vertices $a, b \in \Gamma_0$. We define the graph $\Gamma^1(a, b)$ as follows.*

- *A vertex is a path from a to b in Γ_0.*
- *An edge from path p to path q is a triple (g, α, f). Here f, g are paths in Γ_0 and $\alpha \in \Gamma(t(f), s(g))$ and $g * s(\alpha) * f = p$ and $g * t(\alpha) * f = q$. The data can be visualized as*



We define another graph $\Gamma^2(a, b)$

- *A vertex is a path from a to b in Γ_0.*
- *An edge from path p to path q is a tuple (h, β, g, α, f). Here f, g, h are paths in Γ_0 and $\alpha \in \Gamma(t(f), s(g))$ and $\beta \in \Gamma(t(g), s(h))$ and $h * s(\beta) * g * s(\alpha) * f = p$ and $h * s(\beta) * g * t(\alpha) * f = q$. The data can be visualized as*



Write $F : \mathbf{grph} \to \mathbf{cat}$ for the free category functor. Define functors

$$F\Gamma^2(a,b) \underset{\Psi}{\overset{\Phi}{\rightrightarrows}} F\Gamma^1(a,b)$$

which are identity on objects and on edges are induced by

$$\Phi(h,\beta,g,\alpha,f) = (h,\beta,g*t(\alpha)*f)\circ(h*s(\beta)*g,\alpha,f)$$

and

$$\Psi(h,\beta,g,\alpha,f) = (h*t(\beta)*g,\alpha,f)\circ(h,\beta,g*s(\alpha)*f).$$

Define $F\Gamma(a,b)$ to be category obtained as the coequalizer

$$F\Gamma^2(a,b) \underset{\Psi}{\overset{\Phi}{\rightrightarrows}} F\Gamma^1(a,b) \longrightarrow F\Gamma(a,b)$$

The objects are the same as in $F\Gamma^1(a,b)$ and the morphisms are equivalence classes generated by the relation $\Phi(f)\sim\Psi(f)$ and composition for any morphism $f\in\Gamma^2(a,b)$.

2.8. Definition. *Fix a computad Γ. We define a 2-category $F\Gamma$ as follows.*

- *The objects are the vertices of Γ_0.*
- *If a,b are objects, the hom-category from a to b is $F\Gamma(a,b)$.*
- *If a,b,c are objects, the composition functor $F\Gamma(b,c)\times F\Gamma(a,b)\to F\Gamma(a,c)$ is concatenation of paths on objects and is given on morphisms by*

$$((k,\beta,h),(g,\alpha,f))\mapsto\Phi(k,\beta,h*g,\alpha,f).$$

One checks that this is a well defined 2-category. Moreover, the construction extends to a functor $F : \mathbf{cmptd}\to 2\mathbf{cat}$.

2.9. Theorem. [Str76] *The functor $F : \mathbf{cmptd}\to 2\mathbf{cat}$ is left adjoint to $U : 2\mathbf{cat}\to \mathbf{cmptd}$.*

2.10. Remark. Consider the computad Γ from Example 2.5 and some 2-category $\mathcal{C}$. The utility of Theorem 2.9 is that a 2-functor $F\Gamma\to\mathcal{C}$ is the same as a computad morphism $\Gamma\to U\mathcal{C}$. This is specified by

- objects a,b,a',b' in $\mathcal{C}$,
- morphisms $f:a\to b$ and $s:a\to a'$ and $u:b\to a'$ and $t:b\to b'$ and $f':a'\to b'$ in $\mathcal{C}$, and
- 2-morphisms $\eta : s\Rightarrow u*f$ and $\epsilon : f'*u\Rightarrow t$ in $\mathcal{C}$.

This is the sort of finite amount of data specifying a Γ shaped pasting diagram in $\mathcal{C}$ which satisfies goal (G1). Inspection of Γ reveals that there is exactly one 2-morphism $(f', s) \Rightarrow (t, f)$ in $F\Gamma$. Call this α_Γ. Now, given a functor $\Phi : F\Gamma \to \mathcal{C}$, one obtains $\Phi(\alpha_\Gamma)$ exactly as one obtains the composite of the pasting diagram from Section 1. The "simple pieces" can be precisely described as the image under Φ of the morphisms generating the 2-morphisms from (f', s) to (t, f).

3. Pasting Diagrams

In this section, we define pasting scheme and pasting diagram. A pasting diagram is essentially a computad Γ equipped with distinguished objects a, b and paths p, q from a to b in Γ_0 such that $\mathrm{Hom}_{F\Gamma}(p, q)$ is a singleton.

3.1. Definition. *Fix a computad Γ.*

- *We say that Γ is **acyclic** when Γ_0 is.*
- *We say that Γ is **planar** when for any 2-edge α, the source and target paths $s(\alpha)$ and $t(\alpha)$ are disjoint. Moreover, each edge appears in at most one $s(\alpha)$ and at most one $t(\alpha)$ with α a 2-edge.*
- *We say that Γ is **non-trivial** when no path appearing as a vertex in any $\Gamma(a, b)$ is empty.*
- *We say Γ is **2-linear** between vertices $a, b \in \Gamma_0$ when given a path $(\phi_n, ..., \phi_1)$ in $\Gamma^1(a, b)$ if e is an edge in $s(\phi_i)$ but not $t(\phi_i)$ for some $1 \leq i < n$ then e is not in any $t(\phi_j)$ for $j \geq i$.*

*We say that Γ is **simple** when it is acyclic, planar, and non-trivial.*

3.2. Lemma. *If Γ is a computad which is 2-linear between vertices a and b, then $\Gamma^1(a, b)$ is acyclic.*

3.3. Notation. *Let Γ be an acyclic computad. If w is a path in Γ_0 and α is a 2-edge of Γ with $s(\alpha) \subseteq w$, define $\alpha_* w$ to be that path in Γ_0 obtained in the following way:*

- *Write $w = w^r * s(\alpha) * w^\ell$.*
- *Define $\alpha_* w = w^r * t(\alpha) * w^\ell$.*

Further, define $\rho_{w,\alpha} : w \to \alpha_* w$ to be the edge (w^r, α, w^ℓ) in $\Gamma^1(s(w), t(w))$.

3.4. Lemma. [Path Regularity] *Fix a simple computad Γ which is 2-linear between a and b. Suppose $\phi = (\phi_n, ..., \phi_1)$ and $\psi = (\psi_m, ..., \psi_1)$ are two paths in $\Gamma^1(a, b)$. If these paths have the same source and the same target, then*

(a) there holds $n = m$,

(b) and there exist 2-edges $\alpha_1, ..., \alpha_n$, paths $w_1, v_1, ..., w_n, v_n$ from a to b, and a permutation σ of $\{1, ..., n\}$ such that $\phi_i = \rho_{w_i,\alpha_i}$ and $\psi_i = \rho_{v_{\sigma(i)},\alpha_{\sigma(i)}}$.

Proof. We consider the path ϕ. By definition of edges in $\Gamma^1(a,b)$ there exist 2-edges $\alpha_1, ..., \alpha_n$ and paths $w_1, ..., w_n$ from a to b such that $\phi_i = \rho_{w_i,\alpha_i}$. Note that 2-linearity and non-triviality implies that the $\alpha_1, ..., \alpha_n$ are distinct. Choose edges $e_i \in s(\alpha_i)$ which is non-empty by non-triviality. By 2-linearity, e_1 is not in the target $t(\phi) = t(\psi)$. But, by planarity, the only 2-edge containing e in its source is α_1. So, some ρ_{v_1,α_1} appears in ψ for some path v_1 from a to b. If $n = 1$, then $n \leq m$. Else, repeat the above analysis with e_2. This process terminates at the n-th step, whence $n \leq m$ and for each $i \in \{1, ..., n\}$ we have some ρ_{v_i,α_i} as an edge in ψ. By an identical argument applies to ψ, we obtain $m \leq n$. So, $m = n$ and all edges in ψ are of the form ρ_{v_i,α_i}. ■

3.5. Theorem. [Path Equivalence] *Fix a simple computad Γ which is 2-linear between a and b. Suppose $\phi = (\phi_n, ..., \phi_1)$ and $\psi = (\psi_n, ..., \phi_n)$ are two paths in $\Gamma^1(a,b)$. If these paths have the same source and the same target, they are equivalent under the relation yielding $F\Gamma^1(a,b)$.*

Proof. We induct on n. The base case of $n = 1$ is clear by planarity. The paths must actually be equal. Write p for the source path of ϕ and ψ. If $n = 2$ and if $\phi \neq \psi$ then $\phi = (\rho_{w,\beta}, \rho_{p,\alpha})$ and $\psi = (\rho_{v,\alpha}, \rho_{p,\beta})$ by path regularity. It follows that (up to the ordering of $s(\alpha)$ and $s(\beta)$) that p decomposes as concatenation

$$p = p^r * s(\beta) * p^m * s(\beta) * p^\ell.$$

At once, the equivalence of ϕ and ψ follows from the definition of the relation defining $F\Gamma(a,b)$.

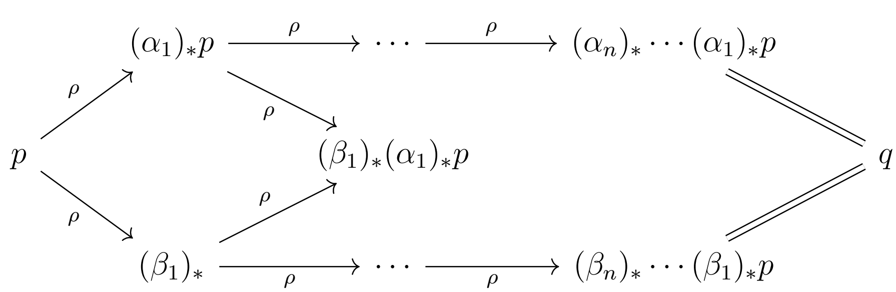
Now, let us assume that $n \geq 2$ and assume the result for shorter paths. If $\phi_1 = \psi_1$, reduce to the $n-1$ case. We will assume this does not occur. Write $\phi_i = \rho_{\bullet,\alpha_i}$ and $\psi_i = \rho_{\bullet,\beta_i}$ where here $\bullet$ is just some path we need not name. Note that the collection $\{\beta_i : i\}$ is the same as $\{\alpha_i : i\}$. With this notation, the path ϕ is

$$p \xrightarrow{\rho} (\alpha_1)_* p \xrightarrow{\rho} \cdots \xrightarrow{\rho} (\alpha_n)_* \cdots (\alpha_1)_* p$$

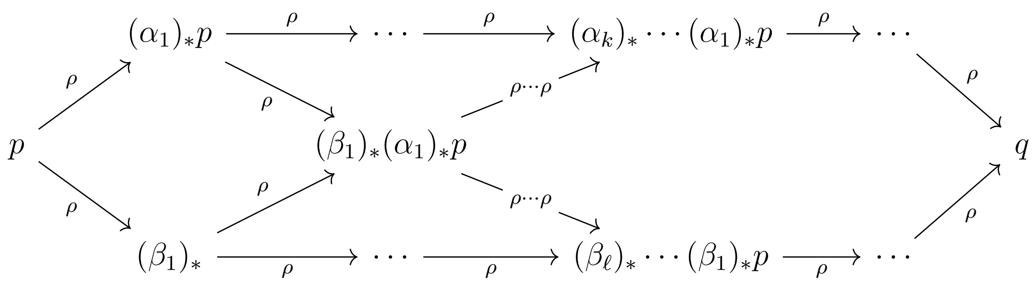
We note that $s(\beta_1)$ is present as a subpath of p. None of these edges are removed by α_1 since $\alpha_1 \neq \beta_1$ from $\phi_1 \neq \psi_1$ and thus are present in $(\alpha_1)_* p$. We can thus form

$$(\beta_1)_*(\alpha_1)_* p = (\alpha_1)_*(\beta_1)_* p.$$

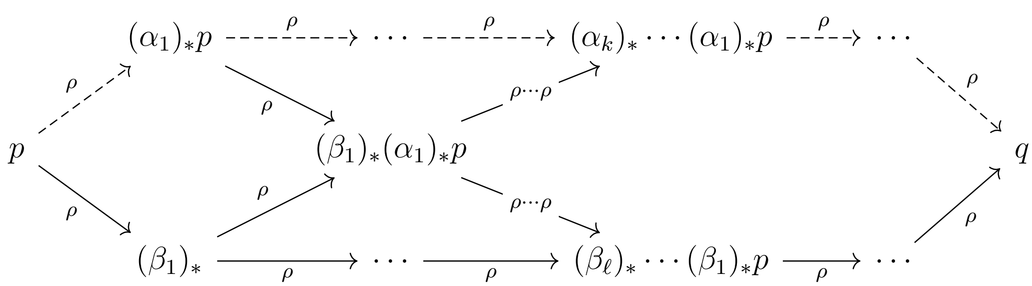
We have the diagram



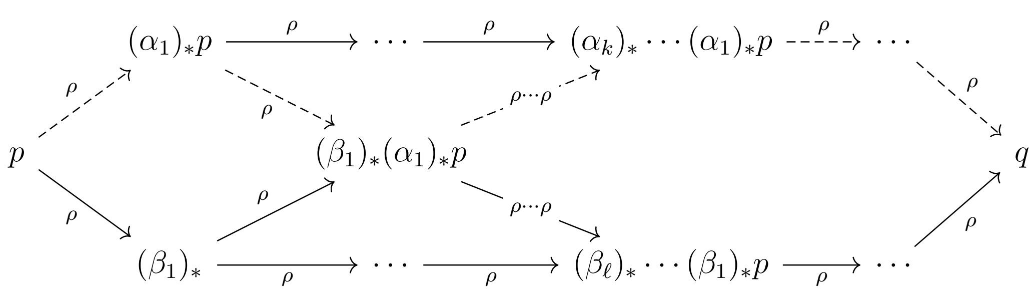
Consider α_2. The path $s(\alpha_2)$ occurs in $(\alpha_1)_*p$. Further, $\alpha_2 \in \{\beta_i\}$. So, unless $\beta_1 = \alpha_2$, we can form $(\alpha_2)_*(\beta_1)_*(\alpha_1)_*p$. Repeat for α_3. So, writing $\beta_1 = \alpha_k$ and $\alpha_1 = \beta_\ell$, we have the diagram



Now, ϕ is the dashed composite



By the induction hypothesis, this is equivalent to the dashed composite

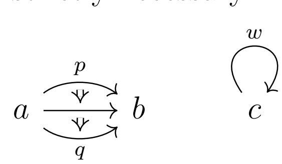


Continuing in this manner we get equivalence with ψ, the bottom path. ■

3.6. Corollary. *If Γ is a simple computad which is 2-linear between a and b, then for any paths p, q from a to b, the hom-set $\mathrm{Hom}_{F\Gamma}(p, q)$ is a singleton.*

3.7. Definition. *A* ***pasting scheme*** *is a triple (Γ, p, q) in which Γ is a simple computad, p, q are paths in Γ_0 from a to b, and Γ is 2-linear from a to b. We write $\alpha_\Gamma \in \mathrm{Hom}_{F\Gamma}(p, q)$ for the unique 2-morphism. If $\mathcal{C}$ is a 2-category, a* ***pasting diagram*** *of shape (Γ, p, q) in $\mathcal{C}$ is a functor $\Phi : F\Gamma \to \mathcal{C}$. Its* ***composite*** *is $\Phi(\alpha_\Gamma)$.*

3.8. Remark. The above definition of pasting scheme is designed so that α_Γ is unique. As presented, it is more rigid than is strictly necessary. For instance, consider the computad



which is not acyclic and thus cannot be the computad underlying a pasting scheme. However, there is clearly a unique morphism in $F\Gamma$ from p to q. The offending loop w is irrelevant. To encode this, one could define the **envelope** $\mathcal{E}_{p,q} \subseteq \Gamma$ to be the smallest subcomputad containing all paths in $\Gamma^2(a,b)$ from p to q. Then one could say that (Γ, p, q) is a pasting scheme when $\mathcal{E}_{p,q}$ is simple and 2-linear from a to b.

4. Comparison with Classical Pasting Schemes

In this section, we review the plane graph based definition of pasting scheme and diagram from [JY21]. We show that any pasting diagram in this sense gives rise to a pasting diagram in the sense of the previous section and that these have the same composite.

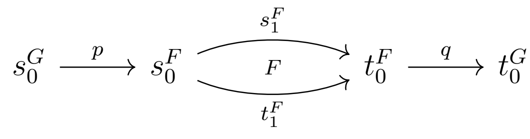
4.1. Definition. *A **plane graph** is a graph G equipped with a specific embedding (of its geometric realization) in the plane. A connected component of $\mathbb{R}^2 \smallsetminus G$ is called **face** of G. The unique unbounded face is called the **exterior face**. The others are **interior faces**. The topological boundary ∂F of a face consists of (the realization of) a set of edges ∂_F of G which is also called the **boundary** of the face.*

4.2. Definition. *If F is a face of a plane graph G, an **anchoring** of F is the choice of the following data:*

- *Two vertices s_0^F and t_0^F called the **vertex source** and **vertex target** of the face;*
- *Two paths s_1^F and t_1^F from s_0^F to t_0^F with disjoint sets of edges each of which belongs to ∂_F. These are called the **path source** and **path target** of the face. When F is the exterior face, one writes $s_i^F = s_i^G$ and $t_i^F = t_i^G$ for $i = 0, 1$ and speaks of the source/target of the graph.*
- *For interior F we make the following additional requirement. While traversing s_1^F, the face F must always lie to the right. While traversing t_1^F, the face F must always lie to the left.*
- *For the exterior face we make the following additional requirement. While traversing s_1^G, the exterior must always lie to the left. While traversing t_1^G, the exterior must always lie to the right.*

*An **anchored graph** is a plane graph equipped with an anchoring of each face.*

4.3. Definition. *An **atomic anchored graph** is an anchored graph with exactly one interior face. All such can be schematically drawn as*



for some paths p, q which may be empty.

4.4. Definition. *Given two anchored graphs G and H with $t_1^G = s_1^H$, there is* ***vertical composite*** *HG. Its underlying graph is the graph theoretic pushout $H \cup_{t_1^G} G$. Its interior faces are (in bijection with) those belonging to either H or G. Each has the same source and target vertices and paths. Further, $s_1^{HG} = s_1^G$ and $t_1^{HG} = t_1^H$. Moreover, $s_0^{HG} = s_0^G = s_0^H$ and $t_0^{HG} = t_0^G = t_0^H$.*

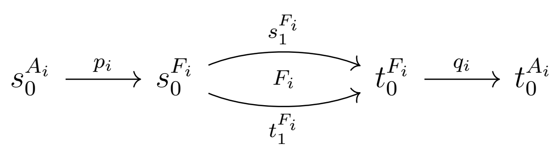
4.5. Definition. *A* ***classical pasting scheme*** *is a finite anchored graph G for which there exists a decomposition $G = A_n A_{n-1} \cdots A_1$ with each A_i atomic.*

4.6. Definition. *Fix a 2-category $\mathcal{C}$ and pasting scheme G. A* ***classical pasting diagram*** *Φ in $\mathcal{C}$ labeled by G consists of the following data.*

- *A functor $\Phi : G \to \mathcal{C}$ from the free category on the digraph underlying G to $\mathcal{C}$.*
- *For each interior face F of G a 2-morphisms $\Phi(F) : \Phi(s_1^F) \to \Phi(t_1^F)$ in $\mathcal{C}$.*

4.7. Definition. *If Φ is a classical pasting diagram in $\mathcal{C}$ labeled by classical pasting scheme G, its* ***composite*** *is that 2-morphisms $\alpha_\Phi : \Phi(s_1^G) \to \Phi(t_1^G)$ obtained by the following procedure.*

(1) Decompose G into atomic pasting schemes $G = A_n A_{n-1} \cdots A_1$. Write each A_i as



(2) For each $i = 1, ..., n$, define $\alpha_i : \Phi(s_1^{A_i}) \to \Phi(t_1^{A_i})$ by whiskering

$$\alpha_i = \Phi(q_i) * \Phi(F_i) * \Phi(p_i).$$

(3) Set $\alpha_\Phi = \alpha_n \alpha_1 \cdots \alpha_1$.

As written, the composite of a pasting diagram depends not just on the diagram but also on the decomposition into atomics. It turns out that the decomposition does not effect the composite.

4.8. Theorem. [JY21] *The composite of a classical pasting diagram is independent of the choice of decomposition into atomics.*

4.9. Definition. *Let G be an anchored graph. We describe now its* ***associated computad*** *Γ^G. The underlying graph $\Gamma_0^G = G$. Given vertices $a, b \in G$, the graph $\Gamma^G(a, b)$ is empty unless there is a face F for which $a = s_0^G$ and $b = t_0^F$. In that case, $\Gamma^G(s_1^F, t_1^F)$ is*

$$s_1^F \xrightarrow{\epsilon_F} t_1^F$$

4.10. Proposition. *If G is a classical pasting scheme, Γ^G is simple and 2-linear from s_0^G to t_0^G.*

Proof. Fix an atomic decomposition $G = A_N \cdots A_1$. We will induct on N. The base case $N = 1$ follows by inspection.

Let us assume that $H = A_{N-1} \cdots A_1$ is simple and 2-linear from s_0^G to t_0^G. We observe that by construction $G = A_N H$ and Γ^G can be obtained from Γ^H as follows. Fix vertices a, b and a path $p : a \to b$ in Γ_0^H. We adjoin to Γ^H a new path $q : a \to b$ and a single 2-edge $\alpha^* : p \to q$. The result is Γ^G.

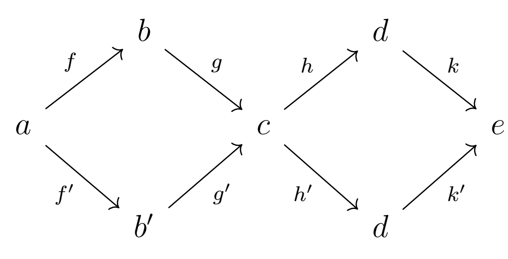
If Γ^G were not acyclic, there would be a cycle in Γ_0^G involving q. In this cycle, replace q by p. The result is a cycle in Γ_0^H which is impossible. So, Γ^G is acyclic.

By construction, Γ^G is planar and non-trivial as soon as Γ^H is.

It only remains to consider 2-linearity. Fix a path $\phi = (\phi_n, ..., \phi_1)$ in $\Gamma^1(s_0^G, t_0^G)$. Suppose e is an edge in $s(\phi_i)$ but not $t(\phi_i)$ for some $1 \leq i < n$. We note that e cannot be in the path q since none of those edges lie in the source of a 2-edge. For each $\phi_j = (g_j, \alpha_j, f_j)$, we observe that if any edge in q lies in the source or target path, then so does the entirety of q. Replace q by p. If $\alpha_j = \alpha^*$, delete ϕ_j. By so doing, obtain a path $\phi' = (\phi'_m, ..., \phi'_1)$. By construction ϕ' is a path in $\Gamma^H(s_0^G, t_0^G)$ and there is $i' = 1, ..., m$ so that e is an edge in $s(\phi'_{i'})$ but not $t(\phi_{i'})$. If there is $k > i$ so that e lies in $t(\phi_k)$, there would then by $k' > i'$ with e in $t(\phi'_{k'})$. This is impossible as Γ^H is 2-linear. So, Γ^G is 2-linear. ∎

4.11. Definition. *When G is a classical pasting scheme, its* ***associated pasting scheme*** *is (Γ^G, s_1^G, t_1^G).*

4.12. Remark. There are pasting schemes which do not arise from classical pasting schemes. For instance, consider the computad Γ whose underlying graph is



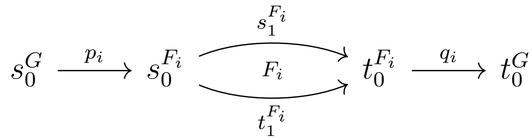
and whose graphs of two edges are empty save for $\Gamma(a, e)$ which contains an unique edge $(k, h, g, f) \to (k', h', g', f')$. Write $\epsilon_i : s_1^{A_i} \to t_1^{A_i}$ for the 2-edge on Γ^G contributed to the face A_i of G. Then write $\phi_i = (q_i, \epsilon_i, p_i)$. Observe that $(\phi_n, ..., \phi_1)$ is a path from s_1^G to t_1^G in Γ^G. So, the composite of the pasting scheme in the sense of this paper is

$$\Phi(q_n * \epsilon_n * p_n) \cdots \Phi(q_1 * \epsilon_1 * p_1).$$

This is precisely the composite of Φ as a classical pasting diagram as defined in Definition 4.7.

4.13. Proposition. *Fix a 2-category $\mathcal{C}$, a classical pasting diagram Φ in $\mathcal{C}$ labeled by G. Observe that Φ is exactly the data of a computad morphism $\Phi : \Gamma^G \to U\mathcal{C}$. The composite of Φ in the classical sense is exactly $\Phi(\alpha_{\Gamma^G})$.*

Proof. Take a decomposition of G into atomic $G = A_n A_{n-1} \cdots A_1$. Write each atomic as



for paths p_i, q_i in G. ■

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