Stacks as 2-Sheaf

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Contents

Let $\mathcal{T}$ be the category of topological spaces and $\mathscr{S}$ be the category of sets. A presheaf $\mathcal{T}^{\mathsf{op}} \xrightarrow{F} \mathscr{S}$ is a sheaf when, for every space X and open cover $\{U_i\}$ of X

$$F(X) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{(i,j)} F(U_{ij})$$

is an equalizer. Here $U_{ij} = U_i \cap U_j$. The goal of this note is to prove that a category $\mathfrak{X}$ fibred in groupoids over $\mathcal{T}$ and viewed as a normal pseudo-functor $\mathcal{T}^{\mathsf{op}} \xrightarrow{\mathfrak{X}} \mathfrak{Grpd}$ is a stack provided for each space X and open cover $\{U_i\}$ of X, the groupoid $\mathfrak{X}(X)$ is the 2-limit of a diagram

$$\prod_i \mathfrak{X}(U_i) \rightrightarrows \prod_{(i,j)} \mathfrak{X}(U_{ij}) \Rrightarrow \prod_{(i,j,k)} \mathfrak{X}(U_{ijk})$$

in the 2-category $\mathfrak{Grpd}$ of groupoids.

The strategy is to prove that the 2-limit of this diagram is exactly the category of descent data for $\mathfrak{X}$ over the cover $\{U_i\}$.

1 A 2-Limit

Notation Let $\Delta^+_{\leq 2}$ denote that subcategory of the simplex category on objects [0], [1], and [2] and whose maps are the monomorphisms. We write $[n] \xrightarrow{u_S} [m]$ for that unique map with image S. Thus, $[0] \xrightarrow{u_1} [1]$ is that map with $u_0(1) = 1$ and $[1] \xrightarrow{u_{0,2}} [2]$ is that map with $u_{0,2}(0) = 0$ and $u_{0,2}(1) = 2$. If $\Delta^+_{\leq 3} \xrightarrow{\mathfrak{X}} \mathcal{C}$ is a pseudo-functor, we will also write $\mathfrak{X}_n \xrightarrow{u} \mathfrak{X}_m$ for the image of $[n] \xrightarrow{u} [m]$ under $\mathfrak{X}$.

Concession to Laziness I'll assume that the pseudo-functor presenting our category fibred in groupoids is an honest functor. This can always be assumed up to equivalence, so there is not much loss.

Construction of $\mathcal{L}$ Fix a functor $\Delta^+_{\leq 3} \xrightarrow{\mathfrak{X}} \mathfrak{Grpd}$. We define a groupoid $\mathcal{L} = \mathcal{L}_{\mathfrak{X}}$ as follow.

- An object is a pair (x, ϕ). Here $x \in \mathfrak{X}_0$ and $u_0(x) \xrightarrow{\phi} u_1(x)$ is a morphism in $\mathfrak{X}_1$. We further require that the diagram

$$u_0(x) \xrightarrow{\mathfrak{X}^{(2)}} u_{0,2}u_0(x) \xrightarrow{u_{0,2}(\phi)} u_{0,2}u_1(x) \xrightarrow{\mathfrak{X}^{(2)}} u_2(x)$$

$$u_0(x) \xrightarrow{\mathfrak{X}^{(2)}} u_{0,1}u_0(x) \xrightarrow{u_{0,1}(\phi)} u_{0,1}u_1(x) \xrightarrow{\mathfrak{X}^{(2)}} u_1 x \xrightarrow{\mathfrak{X}^{(2)}} u_{1,2}u_0(x) \xrightarrow{u_{1,2}(\phi)} u_{1,2}u_1(x) \xrightarrow{\mathfrak{X}^{(2)}} u_2(x)$$

commutes. Call this the cocycle condition. Here $\mathfrak{X}^{(2)}$ is the natural distribution of $\mathfrak{X}$ over function composition (which we suppose is identity).

- A morphism $(x, \phi) \xrightarrow{\alpha} (y, \psi)$ is a morphism $x \xrightarrow{\alpha} y$ in $\mathfrak{X}_0$ such that the diagram

$$\begin{array}{ccc} u_0(x) & \xrightarrow{\phi} & u_1(x) \\ {\scriptstyle u_0(\alpha)}\downarrow & & \downarrow{\scriptstyle u_1(\alpha)} \\ u_0(y) & \xrightarrow[\psi]{} & u_1(y) \end{array}$$

commutes.

The goal of the section is to prove

Theorem 1.0.0 **2-Limit Model**

The groupoid $\mathcal{L} = \mathcal{L}_{\mathfrak{X}}$ is the 2-limit of $\mathfrak{X}$.

Remark Let us recall what this means.

- There is a pseudo-natural transformation $\Delta_{\mathcal{L}} \overset{\lambda}{\Rightarrow} \mathscr{X}$. Here $\mathcal{D}_{\mathcal{L}}$ is the constant functor with value $\mathcal{L}$.
- For any other groupoid $\mathcal{G}$, the induced functor $\mathrm{Hom}(\mathcal{G}, \mathcal{L}) \to \mathrm{Hom}(\Delta_{\mathcal{G}}, \mathfrak{X})$ is an equivalence of categories.

As a first step, we construct λ.

Construction of λ We define a natural transformation $\Delta_{\mathcal{L}} \overset{\lambda}{\Rightarrow} \mathfrak{X}$ as follows.

- The component $\mathcal{L} \xrightarrow{\lambda_0} \mathfrak{X}_0$ is given by $\lambda_0(x, \phi) = x$. The action on morphisms is similar.
- The component $\mathcal{L} \xrightarrow{\lambda_1} \mathfrak{X}_1$ is given by $\lambda_1(x, \phi) = u_0(x)$ and likewise on morphisms.
- The component $\mathcal{L} \xrightarrow{\lambda_2} \mathfrak{X}_2$ is given by $\lambda_2(x) = u_{0,1}u_0(x) = u_0(x)$ and likewise on morphisms.

This specifies the on-objects components of λ. We now give the on-morphism components:

- All identity components are identity.
- The $[0] \xrightarrow{u_0} [1]$ component is identity.
- The component for $[0] \xrightarrow{u_1} [1]$ is

$$\lambda_1(x, \phi) = u_0(x) \xrightarrow{\phi} u_1(x) = u_1\lambda_0(x, \phi).$$

- The component for $[1] \xrightarrow{u_{0,1}} [2]$ is identity.
- The component for $[1] \xrightarrow{u_{0,2}} [2]$ is identity.
- The component for $[1] \xrightarrow{u_{1,2}} [2]$ is

$$\lambda_2(x, \phi) = u_{0,1}u_0(x) \xrightarrow{u_{0,1}(\phi)} u_{0,1}u_1(x) = u_{1,2}\lambda_1(x, \phi).$$

- The component for $[0] \xrightarrow{u_0} [2]$ is identity.
- The component for $[0] \xrightarrow{u_1} [2]$ is

$$\lambda_2(x, \phi) = u_{0,1}u_0(x) \xrightarrow{u_{0,1}(\phi)} u_{0,1}u_1(x) = u_1\lambda_0(x, \phi).$$

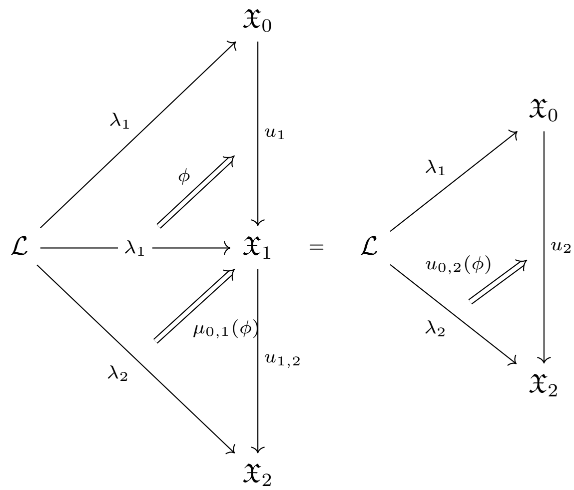
- The component for $[0] \xrightarrow{u_2} [2]$ is

$$\lambda_2(x,\phi) = u_{0,2}u_0(x) \xrightarrow{u_{0,2}(\phi)} u_{0,2}u_1(x) = u_2\lambda_0(x,\phi).$$

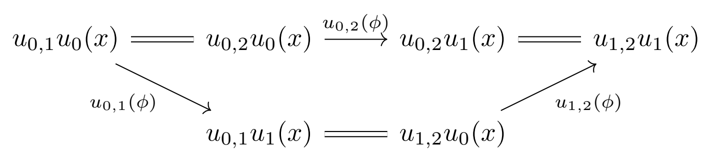
Lemma 1.0.0 — **Well Definition of λ**

The above construction yields a pseudo-natural transformation $\mathcal{D}_{\mathcal{L}} \xrightarrow{\lambda} \mathscr{X}$.

Proof. This is a tedious series of checks best done in the comforts of one's office. The only check which is not immediate is the pasting equality



This may be equivalently phrased for each $(x,\phi) \in \mathcal{L}$ as the commutativity of



This is assured by the definition of $\mathcal{L}$. QED

Lemma 1.0.1 — **Equivalence**

For any groupoid $\mathcal{G}$, the functor $\mathrm{Hom}(\mathcal{G},\mathcal{L}) \xrightarrow{\Phi} \mathrm{Hom}(\Delta_{\mathcal{G}},\mathfrak{X})$ given by pushforward along λ is an equivalence of categories.

Proof. First, let us fix some notation. Suppose that $\mathcal{G} \xrightarrow{F'=(F,\phi)} \mathcal{L}$ is a functor. Then for any object $x \in \mathcal{G}$ we have some $(Fx,\phi_x) \in \mathcal{L}$ where $Fx \in \mathfrak{X}_0$ and $u_0(Fx) \xrightarrow{\phi_x} u_1(Fx)$ is a map. If $x \xrightarrow{f} y$ is a map in $\mathcal{G}$ then $(Fx,\phi_x) \xrightarrow{F'f} (Fy,\phi_y)$ is a map in $\mathcal{L}$ with $Fx \xrightarrow{Ff} Fy$ in $\mathfrak{X}_0$.

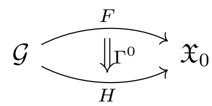
With this setup, we describe $\Delta_{\mathcal{G}} \xrightarrow{\Phi(F')} \mathfrak{X}$. The component of this natural transformation at $i \in \{0,1,2\}$ is exactly the composite $\mathcal{G} \xrightarrow{F} \mathcal{L} \xrightarrow{\lambda_i} \mathfrak{X}$. The component at a map $u \in \Delta^+_{\leq 2}$ is exactly the whiskering $\lambda_u * F$.

Now, suppose that $\mathcal{G} \xrightarrow{H'=(H,\psi)} \mathcal{L}$ is another functor and that $F' \overset{\alpha'}{\Rightarrow} H$ is a natural transformation. For every $x \in \mathcal{G}$ we then have $Fx \xrightarrow{\alpha'_x=\alpha_x} Hx$ in $\mathfrak{X}_0$. This yields a natural transformation $F \overset{\alpha}{\Rightarrow} H$ between functor $\mathcal{G} \xrightarrow{F,H} \mathfrak{X}_0$. So, applying Φ, we obtain a modification $\Phi(F') \overset{\Phi(\alpha')}{\Rightarrow} \Phi(H')$. Its component at $i \in \{0,1,2\}$ is the whiskering $\lambda_i * \alpha$. In particular $\Phi(\alpha')_0 = \alpha$.

We will now prove that Φ is fully faithful and essentially surjective.

The equality $\Phi(\alpha')_0 = \alpha$ above tells us that α' may be recovered from $\Phi(\alpha')$ so that Φ is faithful.

Consider functors $\mathcal{G} \xrightarrow{F',H'} \mathcal{L}$ as above and consider a modification $\Phi(F') \overset{\Gamma}{\Rightarrow} \Phi(H')$. We take the component at $[0] \in \mathcal{D}^+_{\leq 2}$ to obtain a natural transformation

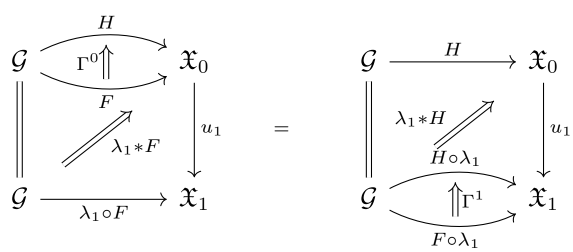


Thus, for each $x \in \mathcal{G}$ we have a map $Fx \xrightarrow{\Gamma^0_x} Hx$ in $\mathfrak{X}_0$. We claim that these are also maps $(Fx, \phi_x) \xrightarrow{\Gamma^0_x} (Hx, \psi_x)$ in $\mathcal{L}$ and that this action of Γ is natural. Once done, this proves that Φ is full.

We verify that Γ^0_x is a map in $\mathcal{L}$ first. For this, we need show that

$$\begin{array}{ccc} u_0(Fx) & \xrightarrow{\phi_x} & u_1(Fx) \\ {\scriptstyle u_0(\Gamma^0_x)}\downarrow & & \downarrow{\scriptstyle u_1(\Gamma^0_x)} \\ u_0(Hx) & \xrightarrow[\psi_x]{} & u_1(Hx) \end{array}$$

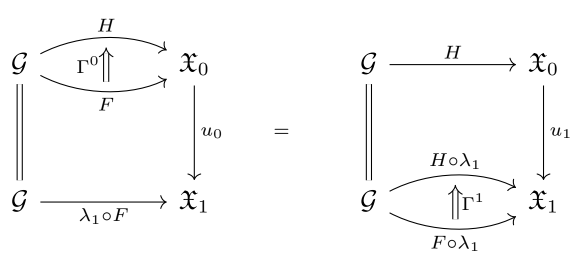
commutes. That Γ is a modification yields the equality.



Reading off a component at $x \in \mathcal{G}$ tells us that

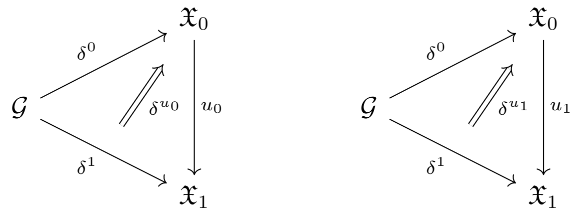
$$\begin{array}{ccc} u_0(Fx) & \xrightarrow{\phi_x} & u_1(Fx) \\ {\scriptstyle \Gamma^1_x}\downarrow & & \downarrow{\scriptstyle u_1(\Gamma^0_x)} \\ u_0(Hx) & \xrightarrow[\psi_x]{} & u_1(Hx) \end{array}$$

commutes. So, we are done once $u_0(\Gamma^0_x) = \Gamma^1_x$. But this follows from the pasting equality

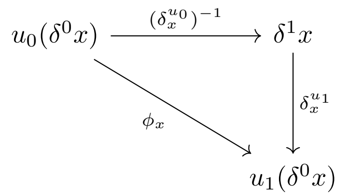


So we have that Γ^0_x lies in $\mathcal{L}$. We must verify the cocycle condition. But this is the same diagram chase as is well definition of λ. Finally, naturality follows at once from naturality of Γ^0 (a map into $\mathcal{L}$ is natural once its projection to $\mathfrak{X}_0$ is).

We have shown that Φ is fully faithful. We must now show that it is essentially surjective. To this end, fix a pseudo-natural transformation $\Delta_{\mathcal{G}} \xrightarrow{\delta} \mathfrak{X}$. We have then a functor $\mathcal{G} \xrightarrow{\delta^0} \mathfrak{X}_0$. We also have diagrams



which yields for each $x \in \mathcal{G}$ a composite map $\delta^1 x \xrightarrow{\delta_x^{\mu_1}} u_0(x)$.



We then define $\mathcal{G} \xrightarrow{D} \mathcal{L}$ by $Dx = (\delta^0 x, \phi_x)$ on objects and by δ^0 on morphisms.

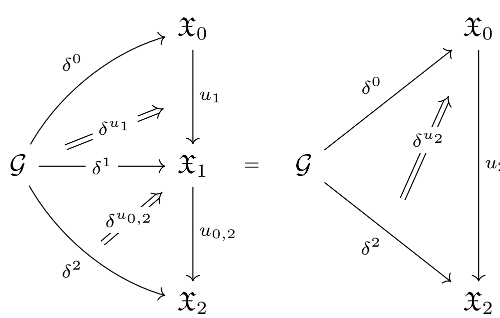
We must now verify that ϕ_x satisfies the cocycle condition so that D is well defined. That is, we must prove $u_{0,1}(\phi_x) = u_{1,2}(\phi_x) u_{0,1}(\phi_x)$. Using the definition of ϕ_x this is the same as

$$u_{0,2}(\delta_x^{u_1}(\delta_x^{u_0})^{-1}) = u_{1,2}(\delta_x^{u_1}(\delta_x^{u_0})^{-1}) u_{0,1}(\delta_x^{u_1}(\delta_x^{u_0})^{-1}).$$

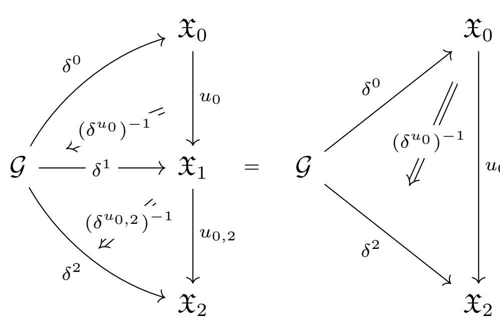
This is equivalent to

$$u_{0,2}(\delta_x^{u_1}) u_{0,2}(\delta_x^{u_0})^{-1} = u_{1,2}(\delta_x^{u_1}) u_{1,2}(\delta_x^{u_0})^{-1} u_{0,1}(\delta_x^{u_1}) u_{0,1}(\delta_x^{u_0})^{-1}.$$

We now use



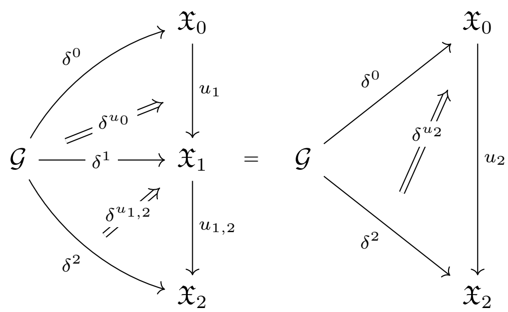
and



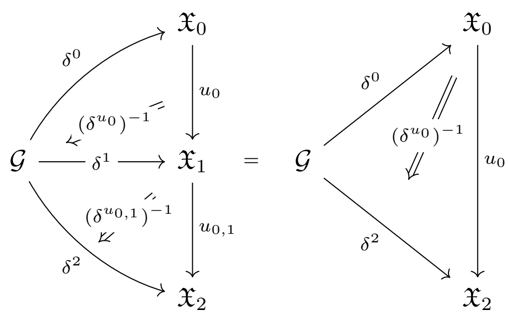
to deduce

$$\begin{aligned} u_{0,2}(\delta_x^{u_1}) u_{0,2}(\delta_x^{u_0})^{-1} &= u_{0,2}(\delta_x^{u_1}) \delta_x^{u_{0,2}} (\delta_x^{u_{0,2}})^{-1} u_{0,2}(\delta_x^{u_0})^{-1} \\ &= \delta_x^{u_2} (\delta_x^{u_0})^{-1} \end{aligned}$$

Next, we use



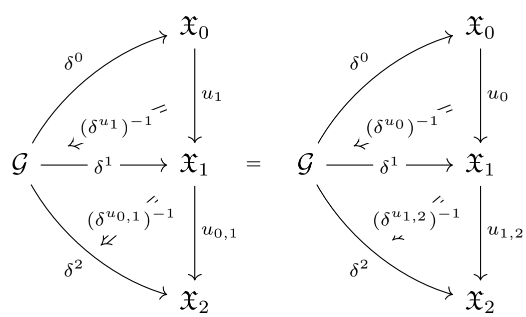
and



to obtain

$$
\begin{aligned}
\delta_x^{u_2}(\delta_x^{u_0})^{-1} &= u_{1,2}(\delta_x^{u_1})\delta_x^{u_{1,2}}(\delta_x^{u_{0,1}})^{-1}u_{0,1}(\delta_x^{u_0})^{-1} \\
&= u_{1,2}(\delta_x^{u_1})\delta_x^{u_{1,2}}(\delta_x^{u_{0,1}})^{-1}u_{0,1}(\delta_x^{u_1})^{-1}u_{0,1}(\delta_x^{u_1})u_{0,1}(\delta_x^{u_0})^{-1} \\
&= u_{1,2}(\delta_x^{u_1})\delta_x^{u_{1,2}}(\delta_x^{u_{1,2}})^{-1}u_{1,2}(\delta_x^{u_0})^{-1}u_{0,1}(\delta_x^{u_1})u_{0,1}(\delta_x^{u_0})^{-1} \\
&= u_{1,2}(\delta_x^{u_1})u_{1,2}(\delta_x^{u_0})^{-1}u_{0,1}(\delta_x^{u_1})u_{0,1}(\delta_x^{u_0})^{-1}.
\end{aligned}
$$

where we have used



This at last proves the cocycle condition. So, D is a well defined functor. It only remains to show that $\Phi(D) \cong \delta$. Once done, Φ is a fully faithful essential surjection whence an equivalence.

We recall that $\Phi(D)_0 \lambda_0 \circ D = \delta^0$. We have that

$$\Phi(D)_1 = \lambda_1 \circ D = u_0 \circ \delta^0.$$

This is isomorphic to δ^1 via δ^{u_0}. Likewise

$$\Phi(D)_1 = \lambda_2 \circ D = u_0 \circ \delta^0.$$

This is isomorphic to δ^1 via δ^{u_0}. One checks that these are the components of a modification. QED

2 Stacks

So, any category $\mathfrak{X}$ fibred in groupoids over $\mathcal{T}$ paired with an open cover $\{U_i\}$ of a space X defines a (normal pseudo-) functor $\Delta^+_{\leq 2} \xrightarrow{\mathfrak{X}'} \mathfrak{Grpd}$ by

$$\mathfrak{X}'_0 = \prod_i \mathfrak{X}(U_i) \quad \text{and} \quad \mathfrak{X}'_1 = \prod_{(i,j)} \mathfrak{X}(U_{ij}) \quad \text{and} \quad \mathfrak{X}'_2 = \prod_{(i,j,k)} \mathfrak{X}(U_{ijk})$$

and an action on morphisms by restriction. By the prior computation, we see that the 2-limit of this diagram is the category of descent data for the cover. Thus, we prove

Theorem 2.0.2 **Stack as 2-Sheaf**

A category $\mathfrak{X}$ fibred in groupoids over $\mathcal{T}$ is a stack if and only if for every open cover of a space X the canonical map $\mathfrak{X}(X) \to \lim^2 \mathfrak{X}'$ is an equivalence.