Berger-Joyal duality and traces I

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Abstract

We give a new proof that the opposite of Joyal's disk category $\mathcal{D}_n$ is Berger's wreath product category $\Theta_n = \Delta \wr \cdots \wr \Delta$. Our techniques continue to apply when the simplex category Δ is replaced by Connes' cyclic category Λ and some other crossed simplicial groups.

1 Introduction

Joyal introduced finite combinatorial n-disks in his study of higher categories [Joy97]. These n-disks fit together to determine a category $\mathcal{D}_n$ which he used to introduce a notion of weak n-categories. In this setting, the image of the Yoneda embedding $\mathcal{D}_n^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(\mathcal{D}_n, \mathbf{Set})$ is a collection of n-categories. Berger found an inductive definition Θ_n for this category and produced equivalences

$$\Theta_n \simeq \mathcal{D}_n^{\mathrm{op}}, \tag{1}$$

see [Ber07]. We call Eqn. (1) Berger-Joyal duality.

The inductive definition of Θ_n is written in terms of a wreath product operation

$$(-) \wr (-) : \mathbf{Cat}_{/\Gamma} \times \mathbf{Cat} \to \mathbf{Cat}$$

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where $\mathbf{C}\textsc{at}_{/\Gamma}$ is the category of pairs $C_\gamma := (C, \gamma)$ where C is a small category and $\gamma : C \to \Gamma$ is a functor from C to Segal's category Γ (see Def. 2.0.3). In terms of this operation, Berger defined Θ_n recursively according to the formulas

$$\begin{aligned} \Theta_1 &:= \Delta \\ \Theta_n &:= \Delta \wr \Theta_{n-1} \qquad \text{for} \quad n > 1 \end{aligned}$$

where Δ is the simplex category.

Since the wreath product is functorial, it is easy to study the outputs of $A \wr X$ as A and X vary. The same is not true for n-disks. This makes direct generalizations of Berger-Joyal duality to other settings a tricky business. For example, in order to replace the simplicial category Δ by Connes' cyclic category Λ one needs to intuit the cyclic analogue of an n-disk; while possible, this kind of ingenuity does not represent an extensible solution.

Theorem 4.0.2 contains a functorial construction of Berger-Joyal duality. In order to solve this problem, we introduce a generalized wreath product operation inspired by Kelly's theory of clubs, see Insp. 3.1.1. If C is a 2-category which has pullbacks and a terminal object $*$ and $T : C \to C$ is a functor then there is a T-wreath product 2-functor

$$\otimes^T : C_{/T(*)} \times C \to C$$

see Def. 3.1.1. If C is the category $\mathbf{C}\textsc{at}$ of small categories then we introduce the pair of functors

$$\Pi : \mathbf{C}\textsc{at} \to \mathbf{C}\textsc{at} \qquad \text{and} \qquad \Pi^{\mathrm{op}} : \mathbf{C}\textsc{at} \to \mathbf{C}\textsc{at}$$

in Def. 3.2.1 and Def. 3.4.1. By construction, the Π-wreath product $\otimes^\Pi$ agrees with Berger's wreath product:

$$(-) \otimes^\Pi (-) \cong (-) \wr (-).$$

The functor Π^{op} is defined to be the functor Π conjugated by the involution $-^{\mathrm{op}}$ of $\mathbf{C}\textsc{at}$ and Thm. 3.4.6 shows that the Π^{op}-wreath product satisfies

$$\nabla \otimes^{\Pi^{\mathrm{op}}} \mathcal{D}_n \simeq \mathcal{D}_{n+1} \qquad \text{for} \quad n \geq 1$$

where $\nabla \cong \Delta^{\mathrm{op}}$ is the interval category. Prop. 3.1.3 shows that the dual pair, Π and Π^{op}, of functors determines an isomorphism between the associated dual pair of wreath products:

$$((-) \wr (-))^{\mathrm{op}} \cong (-)^{\mathrm{op}} \otimes^{\Pi^{\mathrm{op}}} (-)^{\mathrm{op}}.$$

These ideas combine in Thm. 4.0.2 to give a functorial proof of Berger-Joyal duality. The base case is $\mathcal{D}_1 \simeq \Delta^{\mathrm{op}}$. Now, assuming $\Theta_n^{\mathrm{op}} \simeq \mathcal{D}_n$,

$$\begin{aligned}\Theta_{n+1}^{\mathrm{op}} &= (\Delta \wr \Theta_n)^{\mathrm{op}} \\ &= (\Delta \otimes^{\Pi} \Theta_n)^{\mathrm{op}} \\ &\cong \Delta^{\mathrm{op}} \otimes^{\Pi^{\mathrm{op}}} \Theta_n^{\mathrm{op}} \\ &\simeq \nabla \otimes^{\Pi^{\mathrm{op}}} \mathcal{D}_n \\ &\simeq \mathcal{D}_{n+1}.\end{aligned}$$

The remainder of the paper contains a few steps towards applying this duality theorem to the study of n-categories. Our motivation is to use this construction to produce operations on higher categories. Roughly speaking, n-categories are presheaves on Θ_n with some additional structure and, as mentioned above, our version of Berger-Joyal duality allows us to replace a copy of Δ in the definition of Θ_n with a category C. When there is a good choice $j : \Delta \to C$ then our definition admits induction-restriction functors

$$PSh(\Delta \wr \cdots \wr \Delta \wr \cdots \wr \Delta) \leftrightarrows PSh(\Delta \wr \cdots \wr C \wr \cdots \wr \Delta).$$

In good cases, these functors can be used to produce operations on higher categories.

A wreath product involving non-standard C depends on a choice of functor $C \to \Gamma$ and §5 contains a study of these functors. In Berger's work, the functor $\Delta \to \Gamma$ keeps track of the edges in the ordered sets $[n] = \{0 < 1 < 2 < \cdots < n\}$, see Def. 3.3.1, but, as we will see, there are many other examples. Theorem 5.1.3 contains a classification of functors $\Delta \to \Gamma$. As a corollary Berger's functor is characterized as extremal. In Section 5.2, we consider the special case in which $C := \Delta G$ is a crossed simplicial group in the sense of Loday and Fiedorowicz [FL91].

In our last section we examine the important special case of $C := \Lambda$ Connes' cyclic category with preparation towards the sequel [CC25] in which we will introduce a family of trace operations on higher categories.

Notation. We use the symbols $=$ for equality, $\cong$ for isomorphism in an ambient category and $\simeq$ for equivalence in an ambient 2-category. If $\gamma \in \mathcal{C}_{/X}$ is an object of the slice category then the functor $\gamma : A \to X$ will be denoted by A_γ.

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2 Basic definitions

Here we recall the simplex category Δ, the interval category ∇, Segal's category of finite sets Γ and Connes' cyclic category Λ. Then in §2.1, Joyal's categories $\mathcal{D}_n$

of finite combinatorial n-disks are reviewed.

Definition 2.0.1. (Δ, ∇) The *simplex category* Δ is the category with objects

$$[n] := \{0 \leq \cdots \leq n\}$$

and order preserving set maps. The interval category ∇ is the category consisting of the objects $[n]$ with $n > 0$ and maps which preserve both the order and the extrema.

In Joyal's preprint, a duality between the interval category and the simplicial category appears. This is a special case of the duality between disks and Θ_n which is proven in §4. There is an isomorphism of the form below.

Proposition 2.0.2. *([Joy97, §1.1])* $\Delta^{op} \cong \nabla$

Segal introduced the category Γ in [Seg74, Def. 1.1] as a tool for identifying infinite loop spaces. Here we recall this category and a few properties.

Definition 2.0.3. (Γ) Segal's category Γ is the opposite of (the skeleton of) the category of finite pointed sets $\mathbf{FinSet}_*^{\mathrm{op}}$.

An explicit description of the skeleton of Γ has objects of the form $\underline{n} := \{1, \ldots, n\}$. A map $f : \underline{n} \to \underline{l}$ is a set map $f : \underline{n} \to \mathcal{P}(\underline{l})$ from $\underline{n}$ to the power set of $\underline{l}$ such that distinct elements $a \neq b \in \underline{n}$ are carried to disjoint subsets $f(a) \cap f(b) = \emptyset$ of $\underline{l}$. The composition of two maps $f : \underline{n} \to \underline{l}$ and $g : \underline{l} \to \underline{k}$ is $(gf)(a) := \cup_{b \in f(a)} g(b)$. The identity map $1_{\underline{n}} : \underline{n} \to \underline{n}$ is $a \mapsto \{a\}$ for $a \in \underline{n}$. There is an equivalence $P : \Gamma \xrightarrow{\sim} \mathbf{FinSet}_*^{\mathrm{op}}$. On sets $A \in \mathrm{ob}(\Gamma)$, $P(A) := A \sqcup \{*\}$ and for a map $P(f) : P(B) \to P(A)$

$$P(f)(t) := \begin{cases} s & t \in f(s) \\ * & t \notin f(s) \end{cases}$$

2.1 Finite combinatorial n-disks

In this section we introduce Joyal's category of combinatorial disks.

Definition 2.1.1. $(\mathcal{D}_n)$ An object X in the category $\mathcal{D}_n$ of *finite combinatorial n-disks* $\mathcal{D}_n$, consists of a collection of sets X_k and set maps s_k, t_k and p_k

$$X_0 \underset{t_0}{\overset{s_0}{\rightrightarrows}} \overset{\leftarrow p_1 -}{} X_1 \underset{t_1}{\overset{s_1}{\rightrightarrows}} \overset{\leftarrow p_2 -}{} X_2 \underset{t_2}{\overset{s_2}{\rightrightarrows}} \overset{\leftarrow p_3 -}{} \cdots \underset{t_{n-1}}{\overset{s_{n-1}}{\rightrightarrows}} \overset{\leftarrow p_n -}{} X_n$$

which satisfy the relations

$$p_k s_{k-1} = 1_{k-1} = p_k t_{k-1}, \quad s_k s_{k-1} = t_k s_{k-1}, \quad s_k t_{k-1} = t_k t_{k-1} \quad \text{for} \quad 1 \leq k \leq n. \quad (2)$$

In addition, we require

(1) $X_0 = \{*\}$ and $s_0(*) \neq t_0(*)$

(2) The equalizer $\mathrm{Eq}(s_k, t_k) = s_{k-1}(X_{k-1}) \cup t_{k-1}(X_{k-1})$ for $1 \leq k < n$

(3) For each $x \in X_k$, the set $p_{k+1}^{-1}(x)$ is a finite linearly order set with minimum $s_k(x)$ and maximum $t_k(x)$.

A map $f : X \to Y$ in $\mathcal{D}_n$ is a collection of maps $f = \{f_k : X_k \to Y_k\}_{k=0}^{n}$ which commute with the maps s_k, t_k and p_k and preserve the linear orders.

Motivation 2.1.2. Just as the collection of topological n-simplices determine a functor $\Delta \to \mathbf{TOP}$, the collection of topological n-disks

$$\mathbb{D}_n := \{x \in \mathbb{R}^n : |x| \leq 1\}$$

determine an object

$$\mathbb{D}_0 \;\begin{array}{c}\xrightarrow{s_0}\\ \xleftarrow{p_1}\\ \xrightarrow[t_0]{}\end{array}\; \mathbb{D}_1 \;\begin{array}{c}\xrightarrow{s_1}\\ \xleftarrow{p_2}\\ \xrightarrow[t_1]{}\end{array}\; \mathbb{D}_2 \;\begin{array}{c}\xrightarrow{s_2}\\ \xleftarrow{p_3}\\ \xrightarrow[t_2]{}\end{array}\; \cdots \;\begin{array}{c}\xrightarrow{s_{n-1}}\\ \xleftarrow{p_n}\\ \xrightarrow[t_{n-1}]{}\end{array}\; \mathbb{D}_n$$

There are projections and inclusions

$$p_k : \mathbb{D}_k \to \mathbb{D}_{k-1} \qquad \text{and} \qquad s_k, t_k : \mathbb{D}_k \to \mathbb{D}_{k+1}.$$

The projection $p_{k+1}(\widehat{x}, x_{k+1}) := \widehat{x}$ maps a vector onto its first k coordinates. The inclusions are given by $s_k(\widehat{x}) := (\widehat{x}, -\sqrt{1 - |\widehat{x}|^2})$ and $t_k(\widehat{x}) := (\widehat{x}, \sqrt{1 - |\widehat{x}|^2})$. The relations in Eqn. (2) above are satisfied by these maps.

Remark 2.1.3. Notice that for an interior point $x \in \mathbb{D}_k$, the fibre $p_{k+1}^{-1}(x) \cong [s_k(x), t_k(x)]$ is a non-degenerate interval with endpoints $s_k(x)$ and $t_k(x)$. On the other hand, for a boundary point $x \in \partial\mathbb{D}_k$, the fibre $p_{k+1}^{-1}(x)$ is trivial. In the definition of a finite combinatorial disks, the topological condition that $y \in \mathbb{D}_k$ is a boundary point is replaced by the condition that y lies in the images of s_k or t_k, see condition (2) in Def. 2.1.1.

3 Wreath products

3.1 Abstract wreath products

We introduce a generalization of Berger's wreath product which depends on a functor $T : \mathcal{C} \to \mathcal{C}$.

Definition 3.1.1. ($\otimes^T$) For a functor $T : \mathcal{C} \to \mathcal{C}$, the *T-wreath product*

$$\otimes^T : \mathcal{C}_{/T(*)} \times \mathcal{C} \to \mathcal{C}$$

is the pullback $A_r \otimes^T X := A \times_{T(*)} T(X)$ in the diagram below.

$$\begin{array}{ccc} A_r \otimes^T X & \xrightarrow{\pi} & T(X) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle T(!_X)} \\ A & \xrightarrow{r} & T(*) \end{array}$$

Here the map $!_X : X \to *$ is the unique map from X to the terminal object $*$.

Remark 3.1.2. (i) Choosing a different model for the pullback perturbs $\otimes^T$ up to natural isomorphism.

(ii) Replacing T by a functor which is naturally isomorphic to T perturbs the wreath product $\otimes^T$ by natural isomorphism.

(iii) If $\phi : A' \to A$ is an equivalence then the induced map $\phi^* \otimes 1_X : A'_{\phi^*(r)} \otimes^T X \to A_r \otimes^T X$ is not necessarily an equivalence. However, when $\mathcal{C} = $ **CAT** and $T = \Pi$ or $T = \Pi^{\mathrm{op}}$ below the maps $T(!_X) : T(X) \to T(*)$ are isofibrations (Prop. 3.2.2 (2) and Prop. 3.4.1 (3)), which implies that the map $\phi^* \otimes 1_X$ is an equivalence.

(iv) In our applications, $T :$ **CAT** $\to$ **CAT** is a 2-functor. This implies that T preserves equivalences of categories.

The proposition below is the reason for studying dual pairs Π and Π^{op} of wreath products.

Proposition 3.1.3. *Let $\mathcal{C}$ be a 2-category and $T : \mathcal{C} \to \mathcal{C}$ is a functor which defines a wreath product $\otimes^T$. If $\tau : \mathcal{C} \to \mathcal{C}$ an automorphism of $\mathcal{C}$ and $T^\tau : \mathcal{C} \to \mathcal{C}$ is the functor defined by $T^\tau = \tau T \tau^{-1}$ then there is a natural isomorphism*

$$\tau\big(A_r \otimes^T B\big) \cong \tau(A_r) \otimes^{T^\tau} \tau(B).$$

Proof. Applying τ to the commutative diagram in Def. 3.1.1 gives the left-hand side below and then applying the relation $\tau T = T^\tau \tau$ gives the right-hand side. The result follows from Rmk. 3.1.2 (i) above.

$$\begin{array}{ccc} \tau(A_r \otimes^T X) & \xrightarrow{\tau(\pi)} & \tau T(X) \\ {\scriptstyle \tau(p)}\downarrow & & \downarrow{\scriptstyle \tau T(!_X)} \\ \tau(A) & \xrightarrow{\tau(r)} & \tau T(*) \end{array} \qquad \begin{array}{ccc} \tau(A_r \otimes^T X) & \xrightarrow{\tau(\pi)} & T^\tau(\tau(X)) \\ {\scriptstyle \tau(p)}\downarrow & & \downarrow{\scriptstyle T^\tau(!_{\tau(X)})} \\ \tau(A) & \xrightarrow{\tau(r)} & T^\tau(*) \end{array}$$

□

Inspiration 3.1.1. (Clubs) For a category C, the category of endomorphisms $[C, C]$ is monoidal under composition. If T is a monad then the overcategory $[C, C]_{/T}$ is also monoidal. When C has pullbacks and T satisfies some conditions, Kelly showed [Kel92] that this monoidal structure descends to a collection of objects in $[C, C]_{/T}$. This is equivalent to $C_{/T(*)}$ via the evaluation at $*$ map.

3.2 Berger's wreath product and Θ_n

In this section, we introduce the wreath product $A \wr B := A \otimes^{\Pi} B$ by setting $T := \Pi$ in the construction from Def. 3.1.1. The product $\otimes^{\Pi}$ determined by the functor Π matches the literature, see [Ber07] or [AH14, Rez10, BR20].

Definition 3.2.1. There is a functor $\Pi : \mathbf{CAT} \to \mathbf{CAT}$. For any category C, there is a category $\Pi(C)$ given by the data below.

(1) An object of $\Pi(C)$ is a pair (I, a) where $I = \{1, 2, \ldots, n\} \in \Gamma$ is a set and $a : I \to \mathrm{ob}(C)$ is a function.

(2) A map $f : (I, a) \to (J, b)$ in $\Pi(C)$ consists of a collection $f = (f_0, \{f_{ji}\})$ where $f_0 : I \to J$ in Γ and for each $i \in I$ and $j \in f_0(i)$, $f_{ji} : a(i) \to b(j)$ is a map in C.

(3) The composition $gf : (I, a) \to (K, c)$ of $f : (I, a) \to (J, b)$ and $g : (J, b) \to (K, c)$ is given by the pair $gf = ((gf)_0, \{(gf)_{ki}\})$ with $(gf)_0 := g_0 f_0$ and $(gf)_{ki} := g_{kj} f_{ji}$ where $j \in f_0(i)$ is the unique element such that $k \in g_0(j)$.

(4) The identity map $1_{(I,a)} : (I, a) \to (I, a)$ is the map $1_{(I,a)} := (1_0, 1_{ji})$ where 1_0 is the identity map $1_0 := 1_I$ on I and $1_{ii} : a(i) \to a(i)$ is the identity map $1_{ii} := 1_{a(i)}$.

If $F : C \to D$ is a functor then there is a functor $\Pi(F) : \Pi(C) \to \Pi(D)$. On objects $\Pi(F)(I, a) := (I, F(a))$. If $f = (f_0, \{f_{ji}\})$ then $\Pi(F)(f) := (f_0, \{F(f_{ji})\})$.

The proposition below gives us some basic properties of Π. In particular, (2) below addresses the isofibration condition mentioned in (iii) of Rmk. 3.1.2.

Proposition 3.2.2. *(1)* $\Pi(*) \cong \Gamma$

(2) If X is a category and $!_X : X \to *$ is the canonical map then the map $\Pi(!_X) : \Pi(X) \to \Pi(*)$ is an isofibration.

Proof. For (1), since $*$ consists of one object o and $End_*(o) = \{1_o\}$, there is one map $triv : I \to \{o\}$. So the objects of $\Pi(*)$ are pairs $(I, triv)$ with $I \in \Gamma$. A map $f : (I, triv) \to (J, triv)$ is a pair $f = (f_0, \{f_{ji}\})$ where the maps $f_{ji} : triv(i) \to triv(j)$ are all required to be identity 1_o.

For (2), isomorphisms in $\mathbf{FinSet}_*$ are permutations so if $f : \Pi(!_X)(I, a) \to J$ is an isomorphism in Γ then $\#f(s) = 1$ is a singleton for all $s \in I$. When $j \in f(i)$ set $b(j) := a(i)$, so there is an isomorphism $\widetilde{f} : (I, a) \to (J, b)$ which satisfies $\Pi(!_X)(\widetilde{f}) = f$ given by $\widetilde{f} = (f, \{f_{ji}\})$ where $f_{ji} := 1_{a(i)} : a(i) \to a(i)$. □

Following Def. 3.1.1, the functor Π leads to a product $\otimes^\Pi$. The definition below contains the details.

Definition 3.2.3. ($\otimes^\Pi$ or $\wr$) The *wreath product* is a 2-functor $\otimes^\Pi : \mathbf{Cat}_{/\Gamma} \times \mathbf{Cat} \to \mathbf{Cat}$. If X and A are categories and $\gamma : X \to \Gamma$ is a functor then $X_\gamma \otimes^\Pi A := X_\gamma \times_\Gamma \Pi(A)$ is the pullback of the diagram below.

$$\begin{array}{ccc} X \otimes^\Pi A & \longrightarrow & \Pi(A) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\gamma} & \Gamma \end{array}$$

More concretely, an object of $X \otimes^\Pi A$ is a pair (I, a) where $I \in X$ is an object of X and $a : \gamma(I) \to \mathrm{ob}(A)$. A morphism $f : (I, a) \to (J, b)$ consists of $f_0 : I \to J$ in X and for each $i \in \gamma(I)$ and $j \in \gamma(f_0)(i)$ a map $f_{ji} : a(i) \to b(j)$ in A. This construction agrees with [Ber07, Def. 3.1], $X \wr A = X \otimes^\Pi A$.

Definition 3.2.4. (Segal map) If $X_\gamma \in \mathbf{Cat}_{/\Gamma}$ is an object then the functor $\gamma : X \to \Gamma$ is called the *Segal map*.

3.3 Wreath products of the form $\Delta \wr C$

We will now give an account of wreath products of the form $\Delta \wr C$. First we need a Segal map $\gamma : \Delta \to \Gamma$.

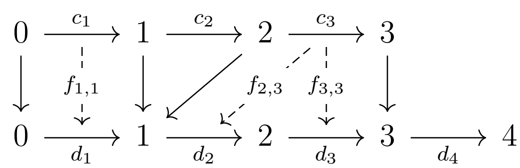
Definition 3.3.1. ($\gamma : \Delta \to \Gamma$) For $[n] \in \Delta$, let $E([n]) := \{e_1, e_2, \ldots, e_n\}$ be the set of edges $e_i : i - 1 \to i$ which generate the poset $[n] = \{0 < 1 < 2 < \cdots < n\}$ as a category. On objects the Segal map is defined by setting $\gamma([n]) := E([n])$.

Now notice that, for each $\phi \in \mathrm{Hom}_{[n]}(i, j)$, there is a subset $E(\phi) \subseteq E([n])$ of edges whose composition is ϕ. If $f : [n] \to [m]$ is a map in Δ and $e \in E([n])$ then the Segal map is given by $\gamma(f)(e) := E(f(e)) \subseteq \gamma([m])$.

Following Def. 3.2.3, the wreath product $\Delta \wr C$ is a category with objects $([n], c)$ where the map $c : E([n]) \to \mathrm{ob}(C)$ labels each edge of $[n]$ by an object $c_i := c(i - 1 \to i)$ of C.

$$0 \xrightarrow{c_1} 1 \xrightarrow{c_2} \cdots \xrightarrow{c_n} n$$

A map $f : ([n], c) \to ([m], d)$ consists of a map $f_0 : [n] \to [m]$ in Δ together with maps $\{f_{ji} : c(i) \to d(j)\}_{j \in f_0(i)}$ in C. The diagram below is a picture of a map $f : ([3], c) \to ([4], d)$



where the vertical solid arrow depicts the morphism f_0 in Δ and the $f_{j,i}$ are morphisms in C. When maps are illustrated in this way, their composition is given by stacking the diagrams and composing those interfacing constituents.

Lastly, we use the wreath product to define Θ_n below.

Definition 3.3.2. (Θ_n) $\Theta_1 := \Delta$ and $\Theta_n := \Delta \wr \Theta_{n-1}$ for $n > 1$.

3.4 Combinatorial disks as wreath products

Following Prop. 3.1.3, there is a conjugate functor

$$\Pi^{\mathrm{op}}(C) := \Pi(C^{\mathrm{op}})^{\mathrm{op}} \tag{3}$$

of Π with respect to the automorphism $-^{\mathrm{op}} : \mathbf{CAT} \to \mathbf{CAT}$. Thm. 3.4.6 below is the opposite $\nabla \otimes^{\Pi^{\mathrm{op}}} \mathcal{D}_n \simeq \mathcal{D}_{n+1}$ of Berger's Θ_n recursion in Def. 3.3.2 above. Before the proving this theorem, the functor Π^{op} is discussed in more detail.

If $(A, *)$ is a pointed set then let $A \backslash * \subset A$ be the result removing the basepoint from the set A.

Proposition 3.4.1. *(1) For a category C, the category $\Pi^{op}(C)$ from Eqn.* (3) *has objects pairs (I, a) with $I \in \Gamma^{op}$ and $a : I \backslash * \to \mathrm{ob}(C)$.*

- *A map $f : (I, a) \to (J, b)$ is a pair $f = (f_0, f_{ji})$ where $f_0 : I \to J$ in Γ^{op} for each $i \in I$ such that $j = f_0(i) \in J \backslash *$, a map $f_{ji} : a(i) \to b(j)$ in C.*
- *If $f : (I, a) \to (J, b)$ and $g : (J, b) \to (K, c)$ then gf is given by $(gf)_0 = g_0 f_0$ and $(gf)_{ki} = g_{kj} f_{ji}$ where $k = g_0(j)$ and $j = f_0(i)$.*

(2) $\Pi^{op}(*) \cong \Gamma^{op}$

(3) If X is a category and $!_X : X \to *$ is the canonical map then the map $\Pi(!_X) : \Pi^{op}(X) \to \Pi^{op}(*)$ is an isofibration.

Proof. Statement (1) comes from unwinding the definitions, while statements (2) and (3) follow from Prop. 3.2.2 and the involution $-^{\mathrm{op}} : \mathbf{CAT} \to \mathbf{CAT}$. In more detail,

(2) $\Pi^{\text{op}}(*) = (\Pi(*^{\text{op}}))^{\text{op}} = (\Pi(*))^{\text{op}} = \Gamma^{\text{op}}$

(3) By Prop. 3.2.2, $\Pi(!_{X^{\text{op}}}) : \Pi(X^{\text{op}}) \to \Gamma$ is an isofibration and $-^{\text{op}}$ automorphism implies $\Pi(!_{X^{\text{op}}})^{\text{op}} : (\Pi(X^{\text{op}}))^{\text{op}} \to \Gamma^{\text{op}} = \Pi^{\text{op}}(X) \to \Pi^{\text{op}}(*)$ is an isofibration.

□

Remark 3.4.2. Berger showed that $\Pi(C)$ as the result of freely closing C under finite products and adding a zero object [Ber07, Lem 3.2]. So the category $\Pi^{\text{op}}(C)$ can be thought of as freely closing C under finite coproducts and adding a zero object. If $a : I\backslash * \to \text{ob}(C)$ identifies a family of objects in C then $(I, a) \in \Pi^{\text{op}}(C)$ is their coproduct.

If $X_\omega \in \mathbf{CAT}_{/\Gamma}$ is an object then $\omega : X \to \Gamma^{\text{op}}$ is called the *coSegal map*. We will write $X \otimes^{\Pi^{\text{op}}} A$ instead of $X_\omega \otimes^{\Pi^{\text{op}}} A$ when it is unambiguous. More unwinding of definitions above produces the definition below.

Definition 3.4.3. The dual wreath product $X_\omega \otimes^{\Pi^{\text{op}}} A := X_\omega \times_{\Gamma^{\text{op}}} A$ is the category with objects given by pairs (x, a) where $x \in X$ and $a : \omega(x)\backslash * \to \text{ob}(A)$ is a function.

- A map $f : (x, a) \to (y, b)$ is a pair $(f_0, \{f_{ji}\})$ where $f_0 : x \to y$ is a map in X and, for each $i \in \omega(x)\backslash *$ such that $\omega(f_0)(i) = j \in \omega(y)\backslash *$, there is a map $f_{ji} : a(i) \to b(j)$ in A.

Remark 3.4.4. We have found two instances of the cowreath product in the literature. For a category C, Borceux introduced a construction $\mathbf{SET}(C)$, the finite analogue $\mathbf{FINSET}(C)$ is the cowreath product $\mathbf{FINSET} \otimes^{\Pi^{\text{op}}} C$ where the coSegal functor $-_+ : \mathbf{FINSET} \to \mathbf{FINSET}_*$ freely adjoints a basepoint, see [Bor94, Ch. 8]. Lurie uses a category $\Delta_S = \Delta_\omega \otimes^{\Pi^{\text{op}}} S$ where S is a set and the coSegal functor $\omega : \Delta \to \mathbf{FINSET}$ is given by $\omega([n]) := \{0, \ldots, n\}$, see [Lur09, Def. 2.1.1].

Proposition 3.4.5. *If $\gamma : \Delta \to \Gamma$ is the Segal functor from Def. 3.3.1 then the coSegal functor $\gamma^{op} : \Delta^{op} \to \Gamma^{op}$ admits a description as $\omega : \nabla \to \mathbf{FINSET}_*$ below under the identifications in Prop. 2.0.2 and Def. 2.0.3.*

- *If $[n] \in \nabla$ then $\omega([n]) := (\{1, 2, \ldots, n-1, *\}, \{*\})$ is the set of non-extreme points.*

- *If $f : [n] \to [m]$ is a map of intervals then $\omega(f) : \omega([n]) \to \omega([m])$ in $\mathbf{FINSET}_*$*

$$\omega(f)(i) := \begin{cases} f(i) & \text{if } f(i) \notin \{0, m\} \\ * & \text{if } f(i) \in \{0, m\} \end{cases}$$

The coSegal map $\omega : \nabla \to \Gamma^{\mathrm{op}}$ allows us to introduce a cowreath product $\nabla \otimes^{\Pi^{\mathrm{op}}} X$ for any category X. The theorem below shows that Joyal's disk categories from Def. 2.1.1 admit an inductive definition in terms of this product.

Theorem 3.4.6. *The disk category is an iterated wreath product,*

$$\nabla \simeq \mathcal{D}_1 \tag{4}$$

$$\nabla_\omega \otimes^{\Pi^{op}} \mathcal{D}_n \simeq \mathcal{D}_{n+1}. \tag{5}$$

Proof. For the base case Eqn. (4), by Def. 2.1.1 an object $X \in \mathcal{D}_1$ is a diagram

$$\{*\} \underset{t_0}{\overset{s_0}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\;p_1\;}}}}} X_1$$

such that $p_1^{-1}(*) = X_1 = \{s_0(*) \leq 1 \leq 2 \leq \cdots \leq t_0(*)\}$ is a finite linearly ordered set with minimum $s_0(*)$ and maximum $t_0(*)$. A map $f : X \to Y$ in $\mathcal{D}_1$ is determined by the order and endpoint preserving map $f_1 : X_1 \to Y_1$. As ∇ is skeleton of the category of finite linearly ordered sets with order and extrema preserving maps, the functor $f : \mathcal{D}_1 \to \nabla$ determined by the assignment $f(X) := [\#X_1]$ is an equivalence of categories.

Now for the induction Eqn. (5), there is a functor

$$\Phi : \mathcal{D}_{n+1} \to \nabla_\omega \otimes^{\Pi^{\mathrm{op}}} \mathcal{D}_n \quad \text{given by} \quad \Phi(X) := (X_1, a) \tag{6}$$

where $a(i) := \tau^i X \in \mathrm{ob}(\mathcal{D}_n)$ assigns to each $i \in \omega(X_1)$ ($i \in X_1$ and i non-extreme) an n-disk $\tau^i X$.

We now check that the definition of Φ makes sense. Recall that an $(n+1)$-disk $X \in \mathcal{D}_{n+1}$ is determined by a collection of data

$$X_0 \underset{t_0}{\overset{s_0}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\;p_1\;}}}}} X_1 \underset{t_1}{\overset{s_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\;p_2\;}}}}} X_2 \underset{t_2}{\overset{s_2}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\;p_3\;}}}}} \cdots \underset{t_{n-1}}{\overset{s_{n-1}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\;p_n\;}}}}} X_n \underset{t_n}{\overset{s_{n-1}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\;p_{n+1}\;}}}}} X_{n+1}$$

satisfying the added conditions in Def. 2.1.1. As before, $X_1 \in \nabla$ is an interval. For each non-extreme element $i \in X_1$, an n-disk $\tau^i X$ can be extracted from the $(n+1)$-disk X by setting

$$\tau^i X_0 := \{i\} \quad \text{and} \quad \tau^i X_k := p_{k+1}^{-1}(\tau^i X_{k-1}) \quad \text{for} \quad 0 < k \leq n$$

in the diagram below.

$$\begin{array}{ccccccccc} \tau^i X_0 & \underset{t_0^i}{\overset{s_0^i}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\;p_1^i\;}}}}} & \tau^i X_1 & \underset{t_1^i}{\overset{s_1^i}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\;p_2^i\;}}}}} & \tau^i X_2 & \underset{t_2^i}{\overset{s_2^i}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\;p_3^i\;}}}}} & \cdots & \underset{t_{n-1}^i}{\overset{s_{n-1}^i}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\;p_n^i\;}}}}} & \tau^i X_n \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & & & \rotatebox{90}{$\subseteq$} \\ X_1 & & X_2 & & X_3 & & & & X_{n+1} \end{array}$$

The structure maps $p_k^i : \tau^i X_k \to \tau^i X_{k-1}$ and $s_k^i, t_k^i : \tau^i X_k \to \tau^i X_{k+1}$ are given by restricting those of X: $p_k^i := p_{k+1}|_{\tau^i X_k}$, $s_k^i := s_{k+1}|_{\tau^i X_k}$ and $t_k^i := t_{k+1}|_{\tau^i X_k}$. The relations in Eqn. (2) of Def. 2.1.1 hold for these assignments because they hold for those of X. By definition $\#\tau^i X_0 = \#\{i\} = 1$, conditions (1), (2) and (3) in Def. 2.1.1 are addressed as follows:

(1) Since $s_1(x) = t_1(x) \Leftrightarrow x \in s_0(*) \cup t_0(*)$, $\mathrm{Eq}(s_0^i, t_0^i) \subset \mathrm{Eq}(s_1, t_1) = \emptyset$ because $i \in X_1$ is an not extreme point.

(2) $\mathrm{Eq}(s_k^i, t_k^i) = \{x \in \tau^i X_k : s_k^i(x) = t_k^i(x)\} = \tau^i X_k \cap \{x \in X_{k+1} : s_{k+1}(x) = t_{k+1}(x)\} = \tau^i X_k \cap (s_k(X_k) \cup t_k(X_k)) = s_{k-1}^i(\tau^i X_{k-1}) \cup t_{k-1}^i(\tau^i X_{k-1})$

(3) If $x \in \tau^i X_{k-1} \subset X_k$ then $(p_k^i)^{-1}(x) = p_{k+1}^{-1}(x) = \{s_k(x) \leq 1 \leq \cdots \leq t_k(x)\} = \{s_{k-1}^i(x) \leq 1 \leq \cdots \leq t_{k-1}^i(x)\}$.

Now if $f : X \to Y$ is a map of combinatorial $(n+1)$-disks then by Def. 2.1.1 $f = \{f_k : X_k \to Y_k\}_{k=0}^{n+1}$ is a collection of maps which commute with the structure maps s_k, t_k, p_k of X and $f_k : p_{k+1}^{-1}(x) \to p_{k+1}^{-1}(f_k(x))$ preserves the linear order for all $0 \leq k \leq n+1$ and for all $x \in X_k$. So that for each $i \in \omega(X_1)$ such that $f_1(i) = j \in \omega(Y_1)$, there is a restriction $\tau^{ji} f := \{f_k|_{\tau^i X}\}_{k=0}^{n} : \tau^i X \to \tau^j Y$. This restriction is functorial, if $f : X \to Y$ and $g : Y \to Z$ are maps of $(n+1)$-disks such that $f_1(i) = j \in \omega(Y_1)$ and $g_1(j) = k \in \omega(Z_1)$ then $\tau^{ki}(gf) = \tau^{kj}(g)\tau^{ji}(f)$.

So we conclude that Eqn. (6) defines a functor $\Phi : \mathcal{D}_{n+1} \to \nabla \otimes^{\Pi^{\mathrm{op}}} \mathcal{D}_n$ which assigns to $(n+1)$-disks X, $\Phi(X) := (X_1, a)$ and maps $f : X \to Y$ between $(n+1)$-disks $\Phi(f) := (\Phi(f)_0, \{\Phi(f)_{ji}\})$ where $\Phi(f)_0 := f_1$ and $\Phi(f)_{ji} := \tau^{ji} f$.

To show that Φ is an equivalence of categories, we prove (1) Φ is fully faithful and (2) Φ is essentially surjective.

(1) Φ is fully faithful. There are mutually inverse maps

$$\alpha : \mathrm{Hom}_{\mathcal{D}_{n+1}}(X, Y) \leftrightarrows \mathrm{Hom}_{\nabla \otimes^{\Pi^{\mathrm{op}}} \mathcal{D}_n}((X_1, a), (Y_1, b)) : \beta$$

between sets of morphisms. For α, $\alpha(f) := \Phi(f) = (\Phi(f)_0, \Phi(f)_{ji}) = (f_1, \{\tau^{ji} f\})$ as discussed above. For β, if $g : (X_1, a) \to (Y_1, b)$ then $g = (g_0, \{g_{ji}\})$, so $\beta(g) : X \to Y$ is defined by $\beta(g) := \{\beta(g)_k : X_k \to Y_k\}_{k=0}^{n+1}$ where $\beta(g)_0(*) := *$, $\beta(g)_1 := g_0$ and, for $k \geq 2$, the map $\beta(g)_k : X_k \to Y_k$ is defined by $\beta(g)_k(x) := (g_{ji})_{k-1}(x)$ for $x \in \tau^i X_{k-1}$ (since $X_k = \sqcup_{i \in \omega(X_1)} \tau^i X_{k-1}$).

On one hand, the composition $\alpha\beta(g) = \alpha(1_*, \beta(g)_0, \{\beta(g)_k\}_{k=2}^{\infty}) = (g_1, \{\tau^{ji}\beta(g)_k\}) = g$ so that $\alpha\beta = 1$. On the other hand, the composition $\beta\alpha(f) = \beta\Phi(f) = \beta(f_0, \{\tau^{ji} f\}) = (1_*, \beta(\Phi(f))_1, \{\beta(\Phi(f))_k\}_{k=2}^{\infty})$ where $\beta(\Phi(f))_1 = f_1$ and $\beta(\Phi(f))_k(x) = (\tau^{ji} f)_{k-1}(x) = f_k(x)$ when $x \in \tau^i X_{k-1}$, again since $X_k = \sqcup_{i \in \omega(X_1)} \tau^i X_{k-1}$, $\beta(\Phi(f))_k = f_k$ so that $\beta\alpha = 1$.

(2) Φ is essentially surjective. Suppose that $X \in \mathcal{D}_{n+1}$ is an $(n+1)$-disk. Then there is an equivalence $\phi : X_1 \xrightarrow{\sim} [\ell]$ for some $\ell \in \mathbb{Z}_{\geq 0}$. By setting $\phi_0 := \phi$ and $\phi_{\phi(i)i} := 1_{\tau^i X} : \tau^i X \to \tau^{\phi(i)} X$, this extends to an equivalence $\widetilde{\phi} := (\phi_0, \phi_{ji}) : (X_1, a) \xrightarrow{\sim} ([\ell], \widetilde{a})$ where $\widetilde{a}(i) := \tau^i X$ for $i \in \omega([\ell]) = \{1, 2, \ldots, \ell - 1\}$.

□

4 Berger-Joyal duality

With the notation for both the wreath product $X \wr A = X \otimes^{\Pi} A$ and the cowreath product $Y \otimes^{\Pi^{\text{op}}} B$ in mind, the statement of Prop. 3.1.3 becomes the corollary below.

Corollary 4.0.1. *For $X_\gamma \in \mathbf{CAT}_{/\Gamma}$ and any category A, there is a natural isomorphism*

$$(X_\gamma \otimes^{\Pi} A)^{op} \cong X^{op}_{\gamma^{op}} \otimes^{\Pi^{op}} A^{op}$$

We now have everything that we need to reprove Berger-Joyal duality.

Theorem 4.0.2. *For each $n \in \mathbb{Z}_{\geq 1}$, there is equivalence $\Theta_n^{op} \simeq \mathcal{D}_n$.*

Proof. The proof is by induction. When $n = 1$, there is an equivalence $\mathcal{D}_1 \simeq \nabla$ by Thm. 3.4.6 so that $\mathcal{D}_1 \simeq \Delta^{\text{op}}$ by Prop. 2.0.2. Now assuming that $\Theta_n^{\text{op}} \simeq \mathcal{D}_n$,

$$\begin{aligned}
\Theta_{n+1}^{\text{op}} &= (\Delta \wr \Theta_n)^{\text{op}} && \text{(Def. 3.3.2)} \\
&\cong \Delta^{\text{op}}_{\gamma^{\text{op}}} \otimes^{\Pi^{\text{op}}} \Theta_n^{\text{op}} && \text{(Cor. 4.0.1)} \\
&\simeq \Delta^{\text{op}}_{\gamma^{\text{op}}} \otimes^{\Pi^{\text{op}}} \mathcal{D}_n && \text{(Induction)} \\
&\simeq \nabla_\omega \otimes^{\Pi^{\text{op}}} \mathcal{D}_n && \text{(Prop. 3.4.5, Rmk. 3.1.2(iii))} \\
&\simeq \mathcal{D}_{n+1} && \text{(Thm. 3.4.6)}
\end{aligned}$$

□

Other perspectives on Berger-Joyal duality can be found in the references [MZ01] and [Our10].

5 Segal functors for locally finite categories

In order to apply the Berger-Joyal duality Theorem 4.0.2 in new settings, it is necessary to view categories C as categories $C \to \Gamma$ over Γ. Thm. 5.1.3 classifies analogues of Berger's Segal functor so that when C is arbitrary, this allows us to motivate the introduction of other Segal functors. In the second part of this section we will give some examples of Segal functors for crossed simplicial groups ΔG. A crossed simplicial group ΔG is a category C which a extension of Δ by a collection of groups.

5.1 A characterization of Berger's Segal functor

We would like to understand Segal functors $C \to \Gamma$ and our problem is that there are too many of them. Recall that a *sieve* S of an object $x \in C$ is a subfunctor $S \subseteq Y_x$ of the Yoneda functor

$$Y_x(y) := \mathrm{Hom}_C(y, x). \tag{7}$$

If C is locally finite then every sieve S determines a functor $S^{\mathrm{op}} : C \to \Gamma$ because $S : C^{\mathrm{op}} \to \textbf{FinSet}$. This does not use the added basepoint $*$ in a non-trivial way (as in Berger's Def. 3.3.1). What we really want is Segal functors which are "like Berger's Segal functor." To achieve this Prop. 5.1.2 constructs Berger's Segal functor from a sieve and Thm. 5.1.3 classifies sieves on Δ. Combining these results shows that Berger's Segal functor arises from the largest proper sieve.

Construction 5.1.1. If $F : C \to \textbf{FinSet}$ is a functor and $S \subseteq F$ a subfunctor then there is a quotient functor $F/S : C \to \textbf{FinSet}_*$ defined as follows.

The functor $F/S : C \to \textbf{FinSet}_*$ is defined on objects $x \in C$ by $(F/S)(x) := (F(x)\backslash S(x)) \sqcup \{*\}$. If $f : x \to y$ is a map in C then $(F/S)(f) : (F/S)(x) \to (F/S)(y)$ is the map given by

$$(F/S)(f)(t) := F(f)(t) \quad \text{when} \quad t \in F(x) \text{ and } F(f)(t) \in (F/S)(y)$$

and $(F/S)(f)(t) := *$ otherwise.

For a sieve S on an object $x \in C$, this construction gives a functor $Y_x/S : C^{\mathrm{op}} \to \textbf{FinSet}_*$ and so determines a functor $(Y_x/S)^{\mathrm{op}} : C \to \Gamma$ as discussed above.

Next Prop. 5.1.2 checks that Berger's Segal functor from Def. 3.3.1 is obtained by the construction in Const. 5.1.1.

In order to state the proposition below we compose Berger's Segal functor $\gamma : \Delta \to \Gamma$ with the equivalence $P : \Gamma \xrightarrow{\sim} \textbf{FinSet}_*^{\mathrm{op}}$ from Def. 2.0.3 above, this gives a functor

$$\gamma' := \gamma \circ P : \Delta \to \textbf{FinSet}_*^{\mathrm{op}}. \tag{8}$$

on objects $[n]$, $\gamma'([n]) = E([n])_+$ consists of the edges of $[n]$ and the basepoint $*$. For a map $f : [n] \to [m]$ in Δ, $\gamma'(f) : \gamma'([m]) \to \gamma'([n])$ maps each $t \in E(f(e))$ to $e \in E([n])_+$ for each edge $e \in E([n])$.

Proposition 5.1.2. *If $S \subseteq Y_{[1]}$ is the sieve on $[1] \in \Delta$ consisting of constant functions (or non-surjective functions) then there is an isomorphism*

$$Y_{[1]}/S \xrightarrow{\sim} \gamma'$$

between the quotient construction above and Berger's Segal functor γ' from Eqn. (8).

Proof. In order to prove the proposition we construct a natural transformation

$$p = \Big\{ p : (Y_{[1]}/S)([n]) \xrightarrow{\sim} \gamma'([n]) \Big\}_{[n]\in \mathrm{ob}(\Delta)}$$

Since there is one edge $e_1 \in \gamma([1])$ and for each edge $e \in \gamma([n])$, there is a unique surjection $p_e : [n] \to [1]$ such that $p_e(e) := e_1$, we introduce maps, $p : \gamma'([n]) \to (Y_{[1]}/S)([n])$ where

$$p(e) := p_e \qquad \text{and} \qquad p(*) := *.$$

Each map p is a bijection because the right-hand side consists of non-constant maps $[n] \to [1]$, $(Y_{[1]}/S)([n]) = \mathrm{Hom}_\Delta([n],[1])\backslash S([n])$ by Construction 5.1.1 and every surjective map $[n] \to [1]$ in the category Δ is equal to p_e for some $e \in \gamma([n])$.

To see that p is natural, for each map $f : [n] \to [m]$ in Δ, we claim that the diagram below commutes, i.e. $f^* \circ p = p \circ \gamma'(f)$.

$$\begin{array}{ccc} \gamma'([m]) & \xrightarrow{p} & (Y_{[1]}/S)([m]) \\ {\scriptstyle \gamma'(f)}\big\downarrow & & \big\downarrow{\scriptstyle f^*} \\ \gamma'([n]) & \xrightarrow[p]{} & (Y_{[1]}/S)([n]) \end{array}$$

On one hand, if $\gamma'(t) = e$ then $p\gamma'(t) = p_e$. On the other hand, $p(t) = p_t : [m] \to [1]$ and $f^*(p(t)) = p_t : [n] \xrightarrow{f} [m] \to [1]$. $\square$

As noted above, any sieve $S \subseteq Y_{[1]}$ determines a Segal functor $(Y_{[1]}/S)^{op} : \Delta \to \Gamma$. The theorem below shows that sieves of the form $S \subseteq Y_{[n]}$ are determined by their images.

Theorem 5.1.3. *Let* $\mathcal{I} := \{S \subseteq Y_{[n]} : S$ *is a sieve on* $[n]\}$ *and let*

$$\mathcal{S} := \{S \subseteq \mathcal{P}([n]) : (A \in S \text{ and } B \subseteq A) \Rightarrow B \in S\}$$

be the set of subsets of $[n] = \{0, 1, \ldots, n\}$ *which are closed under subsets. Then there are mutually inverse maps*

$$\Phi : \mathcal{S} \leftrightarrows \mathcal{I} : \Psi$$

which are determined by the assignments

$$\Phi(S)([k]) := \{f \in \mathrm{Hom}_\Delta([k],[n]) : \mathrm{im}(f) \in S\} \text{ and}$$
$$\Psi(I) := \{S : S = \mathrm{im}(f) \text{ for some } f \in I([k])\}.$$

Proof. The proof consists of four steps. First we show that for $S \in \mathcal{S}$, $\Phi(S) \subseteq Y_{[n]}$ is a sieve on $[n]$. Then we check that for $I \in \mathcal{I}$, $\Psi(I) \in \mathcal{S}$. Lastly, we compute that $\Psi\Phi = 1_{\mathcal{S}}$ and $\Phi\Psi = 1_{\mathcal{I}}$ respectively.

Step 1: If $S \in \mathcal{S}$ then $\Phi(S)$ is a sieve. Since $\Phi(S)([k]) \subseteq Y_{[n]}$, suppose $f : [k] \to [\ell]$ then $\Phi(f) : \Phi(S)([\ell]) \to \Phi(S)[k]$ is pullback $g \mapsto g \circ f$. Φ is closed under pullback because if $\mathrm{im}(g) \in S$ and $\mathrm{im}(g \circ f) \subseteq \mathrm{im}(g)$ then $\mathrm{im}(g \circ f) \in S$.

Step 2: If $I \in \mathcal{I}$ then $\Psi(I) \in \mathcal{S}$. Suppose that $A \in \Psi(I)$ and $B \subseteq A$. Since $A \in \Psi(I)$ there is an $f : [k] \to [n]$ such that $A = \mathrm{im}(f)$. If $j := \# f^{-1}(B)$ then there is a map $g : [j] \to [k]$ in Δ with $\mathrm{im}(g) = B$ and $f \circ g \in I$ because I is a sieve. So $B \in \Psi(I)$ and $\Psi(I) \in \mathcal{S}$.

Step 3: $\Psi\Phi = 1_{\mathcal{S}}$. For $S \in \mathcal{S}$,

$$\begin{aligned}\Psi(\Phi(S)) &= \{S : S = \mathrm{im}(f) \text{ for some } f \in \Phi(S)([k])\} \\ &= \{S : S = \mathrm{im}(f) \text{ for some } f : [k] \to [n] \text{ such that } \mathrm{im}(f) \in S\} \\ &= S\end{aligned}$$

Step 4: $\Phi\Psi = 1_{\mathcal{I}}$. For $I \in \mathcal{I}$,

$$\begin{aligned}\Phi(\Psi(I))([k]) &= \{f \in \mathrm{Hom}_\Delta([k],[n]) : \mathrm{im}(f) \in \Psi(I)\} \\ &= \{f \in \mathrm{Hom}_\Delta([k],[n]) : \mathrm{im}(f) \in \{S : S = \mathrm{im}(g) \text{ for some } g \in I([k])\}\} \\ &= I([k])\end{aligned}$$

□

The following characterization of Berger's Segal functor $\gamma : \Delta \to \Gamma$ from Def. 3.3.1 above follows from combining the classification Thm. 5.1.3 above with Prop 5.1.2.

Corollary 5.1.4. *The sieve $S \subseteq Y_{[1]}$ consisting of constant functions $[n] \to [1]$ corresponds to Berger's Segal functor for Δ. S is the largest proper sieve.*

Proof. Using the previous theorem, $\Psi(S) = \{\emptyset, \{0\}, \{1\}\}$. The only larger sieve corresponds to $\Psi(S) \cup \{\{0,1\}\}$ which is equal to all of $\mathcal{P}([1])$ and so not proper. □

5.2 Segal functors for crossed simplicial groups

In this section we will apply some of our ideas to crossed simplicial groups ΔG. The categories ΔG were introduced by Fiedorowicz and Loday [FL91] and Krasauskas [Kra87]. A crossed simplicial group is an extension of the simplicial category Δ by a groupoid in which the composition is required be of a particularly simple form. For us these categories constitute a concrete family of examples, at least some of which, appear in practice.

Definition 5.2.1. (ΔG) A *crossed simplicial group* ΔG is a category equipped with a functor $i : \Delta \to \Delta G$ such that

(1) i is bijective on objects.

(2) Every map $f : [m] \to [n]$ in ΔG a unique factorization of the form $f = \phi g$ where ϕ is in the image of i and g is an automorphism of $[m]$.

If we set $G_n := \mathrm{Aut}_{\Delta G}([n])$ then the collection $G := \{G_n\}_{n=0}^{\infty}$ forms a simplicial set. A crossed simplicial group ΔG is an extension of the simplicial category by this collection of groups. The unique factorization property ensures that for each map $\phi : [m] \to [n]$ in Δ and $g \in G_n$ there are maps

$$\phi^* : G_n \to G_m \qquad \text{and} \qquad g_* : \mathrm{Hom}_{\Delta G}([m],[n]) \to \mathrm{Hom}_{\Delta G}([m],[n])$$

so that the composition $g\phi$ can be rearranged to $g_*(\phi)\phi^*(g)$. In particular, since any two maps can be written ϕg and $\phi' g'$ their composition in ΔG can be described by

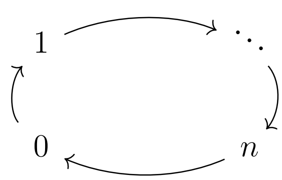
$$(\phi g) \circ_{\Delta G} (\phi' g') = (\phi \circ_{\Delta} g_*(\phi'))(\phi'^*(g) \circ_{G_*} g). \tag{9}$$

a composition in Δ and G_* respectively.

Example 5.2.2. When $G_n = \{1\}$ for all $n \geq 0$ the associated crossed simplicial group $\Delta G = \Delta$ is the simplicial category.

Example 5.2.3. Connes' cyclic category Λ is a crossed simplicial group with $G_n = \mathbb{Z}/(n+1)$. If the set $[n] = \{0, 1, \ldots, n\}$ is viewed as the $(n+1)$-roots of unity in the unit circle $S^1 \subset \mathbb{C}$ then a map $f : [n] \to [m]$ in Λ is a homotopy class of degree 1 map $f : (S^1, [n]) \to (S^1, [m])$.

A combinatorial definition is given using the categories $\langle n \rangle = \{0 < 1 < \cdots < n < 0\}$ generated by the graphs below



If $\omega_i : i \to i$ is the unique map of degree one then a functor $F : \langle n \rangle \to \langle m \rangle$ is called degree one when $F(\omega_i) = \omega_{F(i)}$ for all $i \in \langle n \rangle$. Connes' cyclic category $\Lambda \subset$ **Cat** is equivalent to the subcategory whose objects are $\langle n \rangle$ and whose morphisms are the degree one functors.

Example 5.2.4. The paracyclic crossed simplicial group Λ_∞ has structure groups $G_n = \mathbb{Z}$. If Λ_∞ is the category consisting an object $\widetilde{n} := \mathbb{Z}$ with the standard order for each non-negative integer n then

$$\mathrm{Hom}_{\Lambda_\infty}(\widetilde{n}, \widetilde{m}) := \{f : \mathbb{Z} \to \mathbb{Z} \mid f(l + m + 1) = f(l) + n + 1\}$$

Alternatively, if $\mathbb{Z} = \langle t_{n+1} \rangle$ then $\Delta\mathbb{Z}$ has a presentation with relations

$$\begin{aligned} t_{n+1}\delta_i &= \delta_{i-1}t_n \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad t_{n+1}\delta_0 = \delta_n, \\ t_{n+1}\sigma_i &= \sigma_{i+1}t_{n+2} \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad t_{n+1}\sigma_0 = \sigma_n t_{n+2}^2. \end{aligned}$$

The cyclic category is equivalent to the quotient $\Lambda \cong \Lambda_\infty / \langle t_{n+1}^{n+1} = 1_{[n]} : n \in \mathbb{Z}_{\geq 0} \rangle$.

Example 5.2.5. There is a crossed simplicial group $\Delta\mathbb{Z}/2$ with the same objects as Δ, $\mathrm{ob}(\Delta\mathbb{Z}/2) := \{0 < 1 < \cdots < n\}$ and set maps of the form

$$f : [n] \to [m] \qquad \text{or} \qquad f^* : [n] \to [m]$$

so that

$$\mathrm{Hom}_{\Delta\mathbb{Z}/2}([n],[m]) = \mathrm{Hom}_{\Delta}([n],[m]) \sqcup \mathrm{Hom}_{\Delta}([n],[m])^*$$

such that f preserve the order and f^* preserve the opposite order. There is a special map $y_{n+1} := 1^* : [n] \to [n]$ generating a group $\mathbb{Z}/2$ which can be thought of as order reversing,

$$y_{n+1} \cdot \{0 < 1 < 2 < \cdots < n\} := \{0 > 1 > 2 > \cdots > n\} = \{n < n-1 < \cdots < 1 < 0\}.$$

On maps $\mathrm{Hom}_{\Delta}([n],[m])^* = \mathrm{Hom}_{\Delta}([n],[m]) \cdot y_{n+1}$. These generators y_{n+1} satisfy relations

$$y_{n+1}\delta_i = \delta_{n-i}y_n \qquad \text{and} \qquad y_{n+1}\sigma_i = \sigma_{n-i}y_{n+2}$$

If ΔG is a crossed simplicial group then let

$$Y^G_{[n]}([k]) := \mathrm{Hom}_{\Delta G}([k],[n]) \qquad \text{where} \qquad Y^G_{[n]} : \Delta G^{\mathrm{op}} \to \mathbf{FINSET} \tag{10}$$

be the Yoneda functor on $[n]$ in ΔG (compare to Eqn. (7)). As in §5, by adding basepoints, any sieve $S \subseteq Y^G_{[n]}$ gives a functor $Y^G_{[n]}/S : \Delta G \to \mathbf{FINSET}_*$ so that $(Y^G_{[n]}/S)^{\mathrm{op}} : \Delta G \to \Gamma$ is a candidate Segal functor. In the proposition below we classify sieves of $Y^G_{[n]}$ in terms of sieves on $Y_{[n]}$ since the latter is the content of Thm. 5.1.3 this proposition gives a classification of sieves of $[n]$ in ΔG.

Proposition 5.2.6. *Suppose that $S \subseteq Y_{[n]}$ is a sieve. If $Y^G_{[n]}$ is the Yoneda functor in Eqn.* (10) *above then there is a sieve $S^G \subseteq Y^G_{[n]}$ which consisting of maps $\zeta \in \Delta G$ which factor as $\zeta = \phi g$ with $g \in G_k$ and $\phi \in S([k])$. Moreover, all sieves of $[n]$ in ΔG arise in this way.*

Proof. First we prove that S^G is a sieve. It suffices to show that if $\xi : [\ell] \to [k]$ is a map $\xi \in \Delta G$ then the image of $\xi^* : S^G([k]) \to Y^G([\ell])$ is contained in the subset $S^G([\ell])$. Fix a map $\xi : [\ell] \to [k]$ in ΔG and let $\zeta \in S^G([k])$. The map ξ factors as $\xi = \psi h$ and the map ζ factors as $\zeta = \phi g$. Now, as in Eqn. (9), the pullback factors as

$$\xi^*(\zeta) = \zeta\xi = (\phi \circ g_*\psi) \circ (\psi^* g \circ h).$$

Since S is a sieve on $[n]$ implies that $\phi \circ g_*\psi \in S([\ell])$. So by definition, $\zeta\xi \in S^G$.

Conversely, suppose that $T^G \subseteq Y^G_{[n]}$ is an arbitrary sieve. Define T^Δ to be those elements in T which are morphisms in Δ. As $i : \Delta \to \Delta G$ is faithful, T^Δ is a sieve on $[n]$ in Δ.

We claim that $(T^{\Delta})^G \subseteq T^G$. To see this, fix $f \in (T^{\Delta})^G$. Factor $f = \phi g$ with g an automorphisms and ϕ in Δ. By definition, $\phi \in T^{\Delta} \subseteq T^G$. Thus, since T^G is a sieve, we have $f = \phi g \in T^G$.

On the other hand, since $(T^{\Delta})^G$ is a sieve, we have $T^G \subseteq (T^{\Delta})^G$. Since $T^G = (T^{\Delta})^G$ all sieves ΔG arise in this way. $\square$

Remark 5.2.7. If S is a sieve on $[n] \in \Delta$ then $S^G \cong i_! S$. If S^G is a sieve on $[n] \in \Delta G$ then S^{Δ} is not $i^* S^G$.

Example 5.2.8. Recall from Corollary 5.1.4 that Berger's sieve is the largest proper sieve.

(1) There is a Segal functor on the constant $\mathbb{Z}/2$ crossed simplicial category $\gamma_{\Delta\mathbb{Z}/2} : \Delta\mathbb{Z}/2 \to \Gamma$ which is defined by the quotient $\gamma^{op}_{\Delta\mathbb{Z}/2} := Y^{\mathbb{Z}/2}_{[1]}/B$ where B consists of constant maps $\{0\}$ and $\{1\}$.

(2) There is a Segal functor on the cyclic category $\gamma_{\Lambda} : \Lambda \to \Gamma$ which is defined by the quotient $\gamma^{op}_{\Lambda} := Y_{\langle 0 \rangle}/B$. In this case, there is only one vertex and $B = \emptyset$.

Remark 5.2.9. Example 5.2.8 showcases an interesting phenomenon. Following Ex. 5.2.5 $\Delta\mathbb{Z}/2$ consists of two directed graphs the Segal functor $\gamma_{\Delta\mathbb{Z}/2}$ in (1) above extends $\gamma : \Delta \to \Gamma$ by using these extra edges.

Remark 5.2.10. The opposite of (2) agrees with a cyclic analogue of the Segal functor for Δ in Def. 3.3.1. Let $E(\langle n \rangle)$ be the edges of the graph that generates $\langle n \rangle$ pictured above. If $\phi \in \langle n \rangle$ is a map then write $E(\phi) \subseteq E(\langle n \rangle)$ for the set of elements whose composite is ϕ. The functor $\gamma_{\Lambda} : \Lambda \to \Gamma$ is $\gamma_{\Lambda}(\langle n \rangle) := E(\langle n \rangle)$ on objects. For a map $f : \langle n \rangle \to \langle m \rangle$ and $e \in E(\langle n \rangle)$, set $\gamma_{\Lambda}(f)(e) := E(f(e))$.

Remark 5.2.11. In a different direction, for any diagram $i : \Delta \to C$ then we could define the Segal functor $\gamma_C : C \to \Gamma$ to be the left Kan extension $\gamma_C := \mathrm{Lan}_i \gamma$. It can be shown that this also agrees with the Segal functor for Λ in (2) above.

Once a Segal functor γ_G has been chosen, there is a Berger-Joyal duality for n-fold products of crossed simplicial groups.

Corollary 5.2.12. *If $\Delta G_1, \Delta G_2, \ldots, \Delta G_n$ are crossed simplicial groups equipped with functors $\Delta G_i \to \Gamma$ then there is an equivalence of categories*

$$\Delta G_1 \otimes^{\Pi} \Delta G_2 \otimes^{\Pi} \cdots \otimes^{\Pi} \Delta G_n \cong \Delta G_1^{op} \otimes^{\Pi^{op}} \Delta G_2^{op} \otimes^{\Pi^{op}} \cdots \otimes^{\Pi^{op}} \Delta G_n^{op}$$

References

[AH14] David Ayala and Richard Hepworth. Configuration spaces and Θ_n. *Proc. Amer. Math. Soc.*, 142(7):2243–2254, 2014.

[Ber07] Clemens Berger. Iterated wreath product of the simplex category and iterated loop spaces. *Adv. Math.*, 213(1):230–270, 2007.

[Bor94] Francis Borceux. *Handbook of categorical algebra. 2*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Categories and structures.

[BR20] Julia E. Bergner and Charles Rezk. Comparison of models for (∞, n)-categories, II. *J. Topol.*, 13(4):1554–1581, 2020.

[CC25] Nicholas Cecil and Benjamin Cooper. Berger-Joyal duality and traces II. 2025.

[FL91] Zbigniew Fiedorowicz and Jean-Louis Loday. Crossed simplicial groups and their associated homology. *Trans. Amer. Math. Soc.*, 326(1):57–87, 1991.

[Joy97] André Joyal. Disks Duality and θ-categories. *Preprint*, 1997.

[Kel92] Gregory Maxwell Kelly. On clubs and data-type constructors. volume 177 of *London Math. Soc. Lecture Note Ser.*, pages 163–190. Cambridge Univ. Press, 1992.

[Kra87] Rimvydas Krasauskas. Skew-simplicial groups. *Litovsk. Mat. Sb.*, 27(1):89–99, 1987.

[Lur09] Jacob Lurie. $(\infty, 2)$-categories and the Goodwillie calculus I. *Preprint*, 2009.

[MZ01] Mihaly Makkai and Marek Zawadowski. Duality for simple ω-categories and disks. *Theory Appl. Categ.*, 8:114–243, 2001.

[Our10] David Oury. On the duality between trees and disks. *Theory Appl. Categ.*, 24:No. 16, 418–450, 2010.

[Rez10] Charles Rezk. A Cartesian presentation of weak n-categories. *Geom. Topol.*, 14(1):521–571, 2010.

[Seg74] Graeme Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.