Topology Qual Prep - Practice Qual 1 - 2023

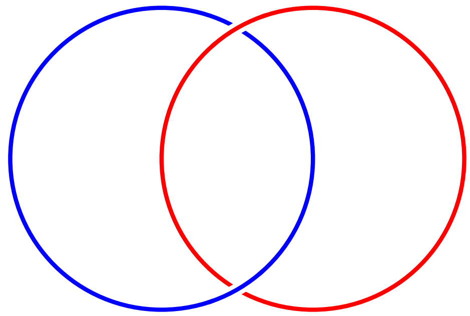
Instructions Do eight problems, four from each part. That is four from part A and four from part B. This is a closed book examination, you should have no books or paper of your own. Please do your work on the paper provided. Clearly number your pages corresponding to the problem you are working. When you start a new problem, start a new page; only write on one side of the paper. Make a cover page and indicate clearly which eight problems you want graded.

Always justify your answers unless explicitly instructed otherwise. You may use theorems if the problem is not a step in proving that theorem, but you need to state any theorems you use carefully.

PART A

Problem 1 Define real projective space and compute the fundamental group of $\mathbb{R}P^2$.

Problem 2 Compute the fundamental group of space obtained by removing a Hopf link from $\mathbb{R}^3$.



Problem 3

(a) Define regular covering. Give two connected coverings of the wedge of two circles, one regular, one not.

(b) If X is a topological space, define what is meant by the suspension $\Sigma(X)$ of X. Prove that if $\pi_1(\Sigma(X))$ is trivial if X is path connected. Does this remain true when X is not path connected?

Problem 4 Let M denote the Möbius band and K the Klein bottle.

(a) Show that M does does not retract onto its boundary.

(b) Show that there exist homotopically non-trivial curves γ, σ in K so that K retracts to σ but not to γ.

(c) Define universal cover. Give a space which is not simply connected which has compact universal cover.

Problem 5 Consider the space Y obtained from $S^2 \times [0,1]$ by identifying $(x,0)$ with $(-x,0)$ and also $(x,1)$ with $(-x,1)$ for all $x \in S^2$.

(a) Show that Y is homeomorphic to the connected sum of $\mathbb{R}P^3$ with itself.

(b) Show that $S^2 \times S^1$ is a double cover of Y.

Problem 6 Let B be a path connected space and $p : E \to B$ a regular n-sheeted cover. Show that if E is path connected then $\pi_1(B)$ has order at least n. Show that if B is simply connected then E has n path components.

PART B

Problem 1 Let M be a smooth manifold equipped with a non-vanishing smooth 1-form α so that $\alpha \wedge d\alpha = 0$. Prove that in a neighborhood U of any point of M there exist smooth functions $f, \lambda : U \to \mathbb{R}$ so that $\alpha = f d\lambda$ on U. You may use the following consequence of the Frobenius Theorem without proof.

> **Corollary** If $\xi \subseteq TM$ is a sub-bundle sections of which are closed under the Lie bracket then any point $p \in M$ has local coordinates (U, x) so that $\xi|_U$ is spanned by $\{\partial_{x_1}, ..., \partial_{x_k}\}$.

Problem 2 Let M be a smooth manifold and $p \in M$. Prove that if $\xi_1, ..., \xi_k$ are linearly independent elements of T_pM then there exists a chart (U, x) about p so that for $i = 1, ..., k$ there holds $\frac{\partial}{\partial x^i}|_p = \xi_i$.

Problem 3 Prove that every Lie group is parallelizable (that is, has trivial tangent bundle).

Problem 4 Let M be a topological manifold with open cover $\mathcal{U}$ and basis $\mathcal{B}$. Prove that there exists a countable, locally finite refinement of $\mathcal{U}$ consisting of elements of $\mathcal{B}$.

Problem 5 Prove that homotopic diffeomorphisms between compact, connected, oriented smooth manifolds are either both orientation preserving or orientation reversing. You may use without proof that homotopic smooth maps are smoothly homotopic.

Problem 6 Consider the function $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ given by $f(x, y) = x^2 + xy + y^2$. Let $S = f^{-1}(\{0\})$.

(a) Show that $S \subseteq \mathbb{R}^2 \setminus \{0\}$ is a regular submanifold.

(b) Show that

$$\xi = y(2\partial_x - \partial_y) + x(\partial_x - 2\partial_y)$$

is a vector field on S.

(c) Let η denote the radial vector field $\eta = x\partial_x + y\partial_y$ on $\mathbb{R}^2$. Prove/Disprove: η is tangent to S. Let $\widetilde{\eta}$ denote the orthogonal projection of η to TS (using the inner product derived from $T\mathbb{R}^2 \cong \mathbb{R}^4$) and compute $\mathscr{L}_\xi \widetilde{\eta}$.

(d) Let $\omega = xdy + ydx$ and compute $\mathscr{L}_\xi \omega$.