Topology Qual Prep - Practice Qual 2 - 2023

Instructions Do eight problems, four from each part. That is four from part A and four from part B. This is a closed book examination, you should have no books or paper of your own. Please do your work on the paper provided. Clearly number your pages corresponding to the problem you are working. When you start a new problem, start a new page; only write on one side of the paper. Make a cover page and indicate clearly which eight problems you want graded.

Always justify your answers unless explicitly instructed otherwise. You may use theorems if the problem is not a step in proving that theorem, but you need to state any theorems you use carefully.

PART A

Problem 1 Let G be a finite group. Show that there exists a smooth manifold M with fundamental group G.

Problem 2 Let M and N be connected topological manifolds of equal positive dimension n.

(a) Define the connected sum $M \# N$.

(b) Assuming $n \geq 3$, find an explicit formula for $\pi_1(M \# N)$ in terms of the fundamental groups of M and N.

(c) Does your formula work for $n = 2$? Justify your answer.

Problem 3 Prove or disprove the following conjecture.

> **Conjecture** If X is a connected, Hausdorff space such that every map from a connected, compact space to X is null homotopic, then X is homotopy equivalent to a point.

Problem 4 If A is a finite subset of a real vector space, then the convex hull of A is the set

$$h(A) = \left\{ \sum_{a \in A} r_a a : r_a \in [0,1] \text{ and } \sum_{a \in A} r_a = 1 \right\}.$$

Let $\{e_1, ..., e_{n+1}\}$ denote the standard basis of $\mathbb{R}^{n+1}$. Then for $n, i \in \mathbb{N}$ and $0 \leq i \leq n$ the standard topological n-simplex and (n,i)-horn are given by

$$|\Delta^n| = h(e_1, ..., e_{n+1}) \text{ and } |\Lambda^n_i| = \bigcup_{j \neq i} h(e_1, ..., \widehat{e_j}, ..., e_n).$$

(a) Draw, as best you are able, the spaces $|\Lambda^2_1|$ and $|\Delta^2|$ and $|\Lambda^3_1|$ and $|\Delta^3|$. (**Aside**: You don't need to show these sitting in some ambient $\mathbb{R}^n$).

(b) Prove that for any topological space X and any continuous map $|\Lambda^n_i| \to X$ there is a continuous extension $|\Delta^n| \to X$.

Problem 5 It is known that if $p : E \to B$ is a covering projection and Y is path connected and locally path connected then any continuous map $f : Y \to B$ lifts to E if $f_* \pi_1(Y)$ is trivial. Show that this result fails if the assumption that Y is locally path connected is removed. (**Hint**: Consider the *Warsaw circle/quasi-circle* which is obtained by connecting the two ends of the topologist's sine curve by a path)

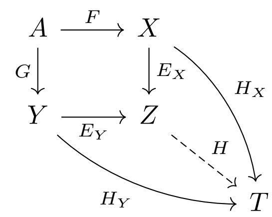
Problem 6 For topological spaces X, Y, write $[X, Y]$ for the set of homotopy classes of maps $X \to Y$. Prove that if A, X, Y are topological spaces and $F \in [A, X]$ and $G \in [A, Y]$ then there exists a topological space Z equipped with $E_X \in [X, Z]$ and $E_Y \in [Y, Z]$ so that the following diagram commutes-up-to-homotopy

$$\begin{array}{ccc} A & \xrightarrow{F} & X \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle E_Z} \\ Y & \xrightarrow[E_Y]{} & Z \end{array}$$

and for any space T and commutative-up-to-homotopy

$$\begin{array}{ccccc} A & \xrightarrow{F} & X & & \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle E_X} & \searrow^{H_X} & \\ Y & \xrightarrow[E_Y]{} & Z & & \downarrow \\ & \searrow_{H_Y} & & \longrightarrow & T \end{array}$$

there exists commutative-up-to-homotopy



PART B

Problem 1 If M is a smooth manifold and $U \subseteq M$ is open, then extension by zero defines a linear map $\iota_U^M : \Omega_c(U) \to \Omega_c(M)$. Suppose that M has open cover $\{U, V\}$.

(a) Does pullback of forms provide a map $\Omega_c(M) \to \Omega_c(U)$?

(b) Prove that there exists a linear map $s : \Omega_c(U) \oplus \Omega_c(V) \to \Omega_c(M)$ so that the following sequence is exact

$$0 \longrightarrow \Omega_c(U \cap V) \xrightarrow{\omega \mapsto \iota_{U\cap V}^U \omega \oplus \iota_{U\cap V}^V(-\omega)} \Omega_c(U) \oplus \Omega_c(V) \xrightarrow{\ s\ } \Omega_c(M) \longrightarrow 0$$

Problem 2 Give a definition of orientation on a manifold. From this definition[1]

(a) prove that if M is a manifold then TM and T^*M are orientable; and

(b) prove that there exist non-orientable manifolds.

Problem 3 Let M be a smooth manifold, $i : S \hookrightarrow M$ an embedded submanifold. Prove that any smooth function $f : S \to \mathbb{R}^k$ may be extended to M.

Problem 4 Consider the map $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^5$

$$f(x, y, z) = (xy, yz, zx, x^2 - y^2, x^2 + y^2 + z^2 - 1).$$

Find an embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$

Problem 5 Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth positive function. Prove that the space obtained by rotating the graph of f about the x-axis is a smooth manifold. Show that this manifold is diffeomorphic to the cylinder $\mathbb{R} \times S^1$.

Problem 6 Let M be a smooth manifold and $f : M \to M$ be a smooth function satisfying $f^2 = f$. Show that $f(M)$ is a submanifold of M.

[1] That is, if you want to use a property/characterization of orientation it must be proven from your given definition.