Nets, Filters, and Convergence

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Abstract

This thesis is an introduction to the study of convergence spaces, a generalization of topological spaces which replaces the primitive notion of open sets with a structure governing the convergence of filters.

Introduction

As one learns math, convergence is encountered in an increasingly general context. Usually, convergence is first encountered in the setting of real, or possibly complex, numbers and is a fairly intuitive idea: a sequence of real numbers converges to a point when it is eventually close to that point no matter how strictly one interprets "close." This makes sense word for word in higher dimensions, and so one obtains the notion of a sequence of points converging to another point in Euclidean $\mathbb{R}^n$.

In fact, $\mathbb{R}^n$ has vastly more structure than is needed for convergence - direction, volume, an algebraic structure, etc. All one needs to formulate something like the convergence seen in the real numbers is a distance notion, and so convergence readily generalizes to any setting equipped with a way to measure distance, *i.e.* a metric space.

In these spaces - $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$, and metric spaces - convergence is dictated by the geometric/topological structure of the space. That is, if one knew enough about the topology on the space, one could answer any question about the convergence of sequences. For instance, if one applies a continuous function to a converging sequence, the result is still a converging sequence. Significantly, the converse is also true; if one knew enough about how sequences behaved in a space, one could reconstruct any desired topological data. For instance, if a function always sends converging sequences to converging sequences (and the limit of the input sequence to the limit of the output sequence), then the function must be continuous.

One might next encounter convergence of sequences in a topological space, a setting much more general than metric spaces. As before, the structure of a topological space informs the convergence of sequences, but here, the other direction breaks down. In general, one cannot use sequences to detect all topological properties. However, all is not lost. One can combat the increased generality of topological spaces by introducing a more general notion of sequence: either filters or nets. These will be defined in Chapter 1 where it will be seen that one can define convergence for nets and filters generalizing the convergence of sequences in a natural manner. The convergence of these objects can then be used to detect topological properties.

There is yet another facet to the story of sequential convergence. There are notions of convergence of sequences common in analysis, *e.g.* convergence almost everywhere in a measure space, which cannot be determined by a notion of distance or topology.

This thesis is an introduction to the study of convergence spaces. Rather than place a structure (*e.g* a distance notion or topology) on a set and allow that structure to determine how sequences, nets, and filters converge, the idea of a convergence space is to state directly what it means for filters to converge. The axioms governing theses spaces generalize the behavior of filters in topological space so that all topological spaces will be convergence spaces. However, they will be general enough to obtain non-topological modes of convergence.

Chapter 1 sets the stage by building up the necessary machinery of filters which is then used throughout later chapters. However, it is the feeling of this author that it is not immediate from the definition how filters generalize sequences; they appear to be an entirely different sort of object. Thus, rather than begin by introducing filters, nets are defined first. These objects are much more sequence-like at first glance. From nets, one can then obtain filters as a sort of compactification of the convergence related properties carried by nets. Indeed, it will be shown that filters and nets are equivalent and interchangeable.

In Chapter 2, convergence spaces will be properly introduced along with a corresponding notion of continuous map. We will see that all topological spaces are convergence spaces and that almost everywhere convergence can also be described as a convergence space. Following this, basic constructions and properties involving convergence spaces will be discussed.

In topology, there is no canonical topology to place on spaces of continuous functions. In Chapter 3, it will be shown that this is not the case for convergence spaces. This chapter explores properties of this canonical convergence structure on function spaces.

Lastly, Chapter 4 discusses convergence vector spaces, the convergence analogue to topological vector spaces. This chapter concludes with a discussion of dual spaces and reflexivity.

In an effort to make this work more self contained, several appendices are included. Appendices A and D addresses basic topics in category theory and topological vector spaces. Appendix C addresses several specific results from general topology which are needed to establish results in the body of the thesis. Appendix B formalizes the equivalence between nets and filters using category theory and then shows how either can be used to define convergence spaces, results which are interesting but somewhat tangential to the body of the text.

Chapter 1

Preliminaries: Nets and Filters

In this chapter, we introduce nets and filters and develop their basic properties and constructions which will be used throughout the rest of the text. Most of the results on filters may be found in Chapter 2.1 of [Pat14] and the results on net-filter equivalence are inspired by the exposition in [Nar].

1.1 A Problem and Its Solution

The convergence of sequences plays an important role in analysis and topology. In fact, the topological properties of metric spaces (continuity, compactness, open and closed sets, etc.) are completely characterized by convergence of sequences. However, this convergence does not fully characterize these topological properties in general. Recall the following

Definition 1.1.1. If X is a topological space, (x_n) a sequence in X, and $x \in X$ then we say (x_n) *converges* to x and write $x_n \to x$ when for each $U \ni x$ open we have some $N \in \mathbb{N}$ so that $x_n \in U$ for all $n \geq N$.

Definition 1.1.2. A mapping of topological spaces $f : X \to Y$ is called *sequentially continuous* when for all $x \in X$ and sequences $x_n \to x$ we have $f(x_n) \to f(x)$ in Y.

Definition 1.1.3. A mapping of topological spaces $f : X \to Y$ is called *continuous* when for each open $U \subseteq Y$ one has that $f^{-1}(U)$ is open in X.

It is well known that all continuous functions are sequentially continuous. However, the converse fails in general.

Example 1.1.4. Let $X = \mathbb{R}$ with the countable-complement topology, that is nonempty $U \subseteq \mathbb{R}$ is open if and only if $X \smallsetminus U$ is finite or countably infinite, and $Y = \mathbb{R}$ with the discrete topology (*i.e.* all sets are open). If (x_n) is a sequence in X converging to a point $x \in X$, then one can see that (x_n) must eventually be constant with value x. Simply consider

$$U = X \smallsetminus \{x_n : n \in \mathbb{N} \ \wedge \ x_n \neq x\}$$

which is an open neighborhood of x. Therefore, any function $f : X \to Y$ is sequentially continuous. However, since Y is discrete and X is not, there must be functions $f : X \to Y$ which fail to be continuous.

More interesting is the fact that even fairly tame notions of sequential convergence can fail to be topological.

Example 1.1.5. Give the interval $[0, 1]$ its usual Lebesgue measure. Let

$$X = \{f : [0, 1] \to [0, 1] : f \text{ is measurable}\}.$$

Ordman shows in [Ord66] that there is not a topology on X so that if (f_n) is a sequence in X and $f \in X$ then $f_n \to f$ in the topology iff $f_n \to f$ almost everywhere.

The first problem, that continuity and sequential continuity are not equivalent and more generally that the convergence of sequences does not characterize topological properties, may be resolved by considering a generalized notion of sequences.

Definition 1.1.6. A *directed set* is a non-empty set I along with a relation $\leq$ which is reflexive, transitive, and such that for each $i, j \in I$ there exists $k \in I$ such that $i, j \leq k$.

Definition 1.1.7. A *net* in a set X is a function $\alpha : I \to X$ where I is a directed set. If $i \in I$, we usually write α_i for $\alpha(i)$. If α is a net, we write $\mathrm{dom}(\alpha)$ for the directed set which is the domain of α.

Since $\mathbb{N}$ with its usual order is a directed set, we see that sequences are merely nets with domain $\mathbb{N}$. Given this, there is a straightforward way to define the convergence of nets in a topological space.

Definition 1.1.8. If α is a net in a set X and $U \subseteq X$, say that α is *eventually in* U and write $\alpha \in_{\text{ev}} U$ when there is some $i_0 \in \mathrm{dom}(\alpha)$ so that for all $i \geq i_0$ we have $\alpha_i \in U$.

Definition 1.1.9. If X is a topological space, α a net in X, and $x \in X$, we say that α *converges to* x and write $\alpha \to x$ when for all neighborhoods U of x we have $\alpha \in_{\text{ev}} U$.

This is a clear generalization of Definition 1.1.2. We may now obtain the following results.

Proposition 1.1.10. *Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if and only if for all $x \in X$ and all nets $\alpha \to x$ we have $f \circ \alpha = f(\alpha) \to x$.*

Proof. Suppose $f : X \to Y$ is continuous, $x \in X$, and α is a net in X with $\alpha \to x$. Suppose U is a neighborhood of $f(x)$. Since f is continuous, we have that $f^{-1}(U)$ is a neighborhood of x. Since $\alpha \to x$, we have that $\alpha \in_{\text{ev}} f^{-1}(U)$. It is then clear that $f(\alpha) \in_{\text{ev}} f(f^{-1}(U)) \subseteq U$. Thus, we have that $f(\alpha) \to f(x)$ as desired.

Now, suppose that for all $x \in X$ and all nets $\alpha \to x$ we have $f \circ \alpha = f(\alpha) \to f(x)$. Suppose $x \in X$ and U is a neighborhood of $f(x)$. Define

$$\mathcal{N}_x = \{V \subseteq X : V \text{ is a neighborhood of } x\}.$$

One may check that when this set is ordered by reverse inclusion of sets it becomes a directed set. Next, define $I = \{(z, V) : z \in V \in \mathcal{N}_x\}$ made into a directed set by considering the ordering on the second coordinate inherited from $\mathcal{N}_x$. Finally, define the net $\alpha : I \to X$ by $\alpha(z, V) = z$ for all $(z, V) \in I$. Note that if W is a neighborhood of x we may pick $z \in W$ and see that for all $(z', W') \geq (z, W)$, we have $z' \in W' \subseteq W$ so that $\alpha(z', W') \in W$. We have that $\alpha \in_{\text{ev}} W$. Therefore, $\alpha \to x$. It follows that $f(\alpha) \to f(x)$. We then have that $f(\alpha) \in_{\text{ev}} U$. From this, $\alpha \in_{\text{ev}} f^{-1}(U)$. We may then find some neighborhood W of x and some $z \in W$ so that for all $(z', W') \in I$ with $W' \subseteq W$ we have $z' \in f^{-1}(U)$. From this, we have that $W \subseteq f^{-1}(U)$ and $f^{-1}(U)$ a neighborhood of x. Therefore, f is continuous as desired. QED

Moreover this result is in no way special. As will be seen, all topological properties are described by convergence of nets in ways analogous to how sequences describe these properties in metric spaces. However, simply swapping nets for sequences cannot fix the issue raised by Example 1.1.4, that some notions of convergence are fundamentally not topological.

Nets do offer a way to resolve this issue. In principal, we will replace the axioms of topological spaces with axioms describing net convergence. Then, topological definitions (of continuity, compactness, etc.) can be defined in such a "convergence space" via their net characterizations from topology. As will be seen, the thereby obtained notion of convergence space will contain the notion of topological spaces but also spaces allowing other notions of convergence, *e.g.* the almost everywhere convergence of Example 1.1.4.

However, there is a slight obstruction to this program. The collection of nets in a non-empty set is not itself a set. To see this, pick a non-empty set X and suppose there is a set N of nets in X. One can then consider the set $\operatorname{dom}(N)$ of directed sets appearing as domains of nets in N. But as any set can be made into a directed set, and any directed set can be mapped to X by the constant net at some point, one sees that $\operatorname{dom}(N)$ contains all sets. This is impossible.

A way to bypass this obstruction arises from the following observation. As far as convergence is concerned, the vast universe of nets is often redundant. For instance, there is no need to distinguish between constant nets with different domains or between eventually constant nets and constant nets. This suggests that it is possible to shrink the class of nets into a well behaved set by identifying together those nets with the same convergence properties. The next section will implement the technical details of this plan.

1.2 Filters

The proof of Proposition 1.1.10 introduces an important object

Definition 1.2.1. If X is a topological space and $x \in X$, we define

$$\mathcal{N}_x = \{U \subseteq X : U \text{ is a neighborhood of } x\}$$

which is called the *neighborhood filter at* x. Note that we do not require neighborhoods to be open.

Indeed, this object controls the convergence of nets in topological spaces.

Definition 1.2.2. If X is a set and α a net in X, define the *eventuality filter* of α by

$$\mathcal{E}(\alpha) = \{U \subseteq X : \alpha \in_{\text{ev}} U\}.$$

Proposition 1.2.3. *If X is a topological space and α a net in X, then for all $x \in X$ we have $\alpha \to x$ if and only if $\mathcal{E}(\alpha) \supseteq \mathcal{N}_x$.*

Proof. Suppose $\alpha \to x$. If $U \in \mathcal{N}_x$, then U contains an open set V containing x. Then $\alpha \in_{\text{ev}} V$ so $\alpha \in_{\text{ev}} U$ and $U \in \mathcal{N}_x$. On the other hand, if $\mathcal{E}(\alpha) \supseteq \mathcal{N}_x$ then $\alpha \in_{\text{ev}} U$ for every open set containing x and $\alpha \to x$ as desired. QED

The eventuality filter of a net and the neighborhood filter of a point are merely instances of a seemingly more general construction.

Notation 1.2.4. If X is a set, we write $\mathcal{P}(X)$ for the powerset of X

Definition 1.2.5. If X is a set, a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is called a *filter* on X when

1. $X \in \mathcal{F}$ but $\emptyset \notin \mathcal{F}$;
2. if $F \in \mathcal{F}$ and $X \supseteq G \supseteq F$ then $G \in \mathcal{F}$;
3. if $F, G \in \mathcal{F}$ then $F \cap G \in \mathcal{F}$.

The set of filters on X will be denoted $\Phi(X)$.

With this definition, it is not hard to verify that the eventuality filter of a net and the neighborhood filter of a point are filters. Beyond this, however, it may not be clear what a filter is, aside from this formal definition. We give some examples of filters.

Example 1.2.6. If X is an infinite set,

$$\mathcal{F} = \{F \subseteq X : X \setminus F \text{ is finite}\}$$

is called the *Frechét filter* on X.

Example 1.2.7. If (X, Σ, μ) is a measure space with $\mu(X) = 1$, then

$$\mathcal{F} = \{F \subseteq X : \exists F' \in \Sigma \text{ such that } F' \subseteq F \ \wedge \ \mu(F') = 1\}$$

is a filter on X.

Proposition 1.2.8. *Suppose X is a set and $\mathcal{B}$ is a collection of subsets of X so that for all $B_1, ..., B_n \in \mathcal{B}$ we have $\bigcap_{i=1}^n B_i \neq \emptyset$, then*

$$[\mathcal{B}] = \left\{ F \subseteq X : \exists B_1, ..., B_n \in \mathcal{B} \text{ such that } F \supseteq \bigcap_{i=1}^{n} B_n \right\}$$

is a filter on X. This is called the filter generated by $\mathcal{B}$*. One calls $\mathcal{B}$ the* base *of* $[\mathcal{B}]$.

Proof. Certainly $X \in [\mathcal{B}]$ and $\emptyset \notin [\mathcal{B}]$ since for all $B_1, ..., B_n \in \mathcal{B}$ we have $\bigcap_{i=1}^n B_n \neq \emptyset$. If $X \supseteq G \supseteq F$ and $F \in [\mathcal{B}]$, there is finite $\mathcal{B}' \subseteq \mathcal{B}$ with $G \supseteq F \supseteq \bigcap \mathcal{B}'$ so that $G \in [\mathcal{B}]$. Lastly, suppose $F, G \in [\mathcal{B}]$. There are then finite $\mathcal{B}', \mathcal{B}'' \subseteq \mathcal{B}$ so that $F \supseteq \bigcap \mathcal{B}'$ and $G \supseteq \bigcap \mathcal{B}''$. We then have that $F \cap G \supseteq \bigcap(\mathcal{B}' \cup \mathcal{B}'')$ so that $F \cap G \in [\mathcal{B}]$. QED

Remark 1.2.9. If $\mathcal{B}$ is closed under finite intersections, then

$$[\mathcal{B}] = \{F \subseteq X : \exists B \in \mathcal{B}(F \supseteq B)\}.$$

Notation 1.2.10. If $B \subseteq X$, we write $[B]$ for $[\{B\}]$, and if $B = \{x\}$ we write $[x]$ for $[B]$. A filter of the form $[x]$ is called a *point filter*.

Despite these examples which seem far removed from nets, any filter on a set may be realized as the eventuality filter of some net.

Proposition 1.2.11. *If X is a set and $\mathcal{F}$ a filter on X, then there exists a net $\eta(\mathcal{F})$ on X so that the eventuality filter of $\eta(\mathcal{F})$ is $\mathcal{F}$.*

Proof. Define

$$I = \{(x, F) \in X \times \mathcal{F} : x \in F\}$$

which we make into a directed set by reverse inclusion on the second coordinate. We then create the net $\eta(\mathcal{F}) : I \to X$ by $\eta(\mathcal{F})(x, F) = x$. One may then verify that $\mathcal{E}(\eta(\mathcal{F})) = \mathcal{F}$. QED

If $\mathcal{F}$ is a filter on X, we call the net constructed in the proof of Proposition 1.2.11 the *derived net* of $\mathcal{F}$. The construction of the derived net is not particularly significant (though it is helpful to have a canonical definition); what matters is that the eventuality filter of the derived net of $\mathcal{F}$ is $\mathcal{F}$. To justify this, note the following result.

Proposition 1.2.12. *If X is a topological space, $x \in X$, and α, β are nets in X with $\mathcal{E}(\alpha) = \mathcal{E}(\beta)$ then $\alpha \to x$ if and only if $\beta \to x$.*

Proof. Suppose $\alpha \to x$. Then $\mathcal{E}(\alpha) \supseteq \mathcal{N}_x$. So $\mathcal{E}(\beta) \supseteq \mathcal{N}_x$ and $\beta \to x$. If $\beta \to x$, an identical argument suffices to finish the proof. QED

Thus, since we care about nets because of their convergence properties, we are entirely justified in neglecting to differentiate between nets with the same eventuality filter. We formalize this.

Definition 1.2.13. If α and β are nets in a set, we say that α and β are *equivalent* and write $\alpha \sim \beta$ when $\mathcal{E}(\alpha) = \mathcal{E}(\beta)$.

So far, we have a way to turn any net into a filter and any filter into a net. A natural question is whether this gives a one-to-one correspondence. Once one applies the above notion of equivalence, the answer is yes.

Theorem 1.2.14. *Fix a set X. Suppose α and β are nets in X and $\mathcal{F}, \mathcal{G}$ are filters on X.*

1. $\mathcal{E}(\eta(\mathcal{F})) = \mathcal{F}$*;*
2. $\eta(\mathcal{E}(\alpha)) \sim \alpha$*.*

Proof. (1) This is merely Proposition 1.2.11.

(2) By (1) we have that $\mathcal{E}(\eta(\mathcal{E}(\alpha))) = \mathcal{E}(\alpha)$ so that $\eta(\mathcal{E}(\alpha)) \sim \alpha$ by definition. QED

Corollary 1.2.15. *If X is a set and $\mathcal{F}, \mathcal{G}$ are filters on X, then $\mathcal{F} = \mathcal{G}$ if and only if $\eta(\mathcal{F}) = \eta(\mathcal{G})$.*

Morally, this means that nets and filters "are the same thing" for the purpose of convergence, an equivalence which is made even more precise in Appendix B.1. Alternatively, one could think of filters as equivalence classes of nets modulo the equivalence of nets given above.

In light of the equivalence between nets and filters, one can use the convergence of nets in a topological space to define the convergence of filters.

Definition 1.2.16. If X is a topological space, $x \in X$, and $\mathcal{F}$ is a filter on X, say that $\mathcal{F}$ *converges to* x and write $\mathcal{F} \to x$ when $\eta(\mathcal{F}) \to x$. By Proposition 1.2.3, we have that $\mathcal{F} \to x$ if and only if $\mathcal{F} \supseteq \mathcal{N}_x$.

The concept of a filter converging, along with the equivalence between nets and filters, bypasses the problems discussed at the end of Section 1.1. The class of nets in a set may not be a set, but convergence only distinguishes between nets up to their eventuality filter, and the collection of filters is a set. Thus, instead of defining convergence spaces by imposing axioms governing net convergence, we will instead place conditions on the convergence of filters.

This section ends with a result on eventually constant nets.

Proposition 1.2.17. *If X is a set and α is a net in X which eventually has value x, then $\mathcal{E}(\alpha) = [x]$ and $\eta([x]) \sim \alpha$.*

Proof. We prove $\mathcal{E}(\alpha) = [x]$ first. Certainly if $x \in F \subseteq X$, then $\alpha \in_{\text{ev}} F$. Thus $\mathcal{E}(\alpha) \supseteq [x]$. Further, if $\alpha \in_{\text{ev}} F \subseteq X$, then it must be that $x \in F$. Thus $\mathcal{E}(\alpha) \subseteq [x]$ and $\mathcal{E}(\alpha) = [x]$ as desired.

We then have by Theorem 1.2.14

$$\alpha \sim \eta(\mathcal{E}(\alpha)) = \eta([x])$$

as desired. QED

Proposition 1.2.18. *If X is a topological space with $x \in X$, then $[x] \to x$.*

Proof. This follows immediately from the fact that $\eta([x])$ is equivalent to a constant net at x. QED

1.3 Subnets

Recall that a metric space is compact if and only if every sequence has a converging subsequence. If there is any hope of extending this result to nets, one must have a notion of subnet. A problem arises in that there are multiple definitions of subnet. The terminology and results on subnets here is largely taken from [Sch97].

Definition 1.3.1. Suppose X is a set and α and β are nets in X, then

1. β is a *subnet in the sense of Aarnes and Andenæs* or simply an *AA subnet* of α when one of the following equivalent conditions holds
   (a) For all $U \subseteq X$ with $\alpha \in_{\text{ev}} U$ one has $\beta \in_{\text{ev}} U$;
   (b) $\mathcal{E}(\beta) \supseteq \mathcal{E}(\alpha)$.
2. β is a *Kelley subnet* of α when there exists $f : \operatorname{dom}(\beta) \to \operatorname{dom}(\alpha)$ so that
   (a) $\beta = \alpha \circ f$;
   (b) f is strongly final, that is for every $a \in \operatorname{dom}(\alpha)$, there exists $b_0 \in \operatorname{dom}(\beta)$ so that $f(b) \geq a$ whenever $b \geq b_0$;
3. β is a *Willard* subnet of α when there exists $f : \operatorname{dom} \beta \to \operatorname{dom} \alpha$ so that
   (a) $\beta = \alpha \circ f$;
   (b) f is monotone, that is for every $a, b \in \operatorname{dom}(\beta)$ with $a \geq b$ there holds $f(a) \geq f(b)$;
   (c) f is final, that is for every $a \in \operatorname{dom}(\alpha)$, there exists $b_0 \in \operatorname{dom}(\beta)$ so that $f(b_0) \geq a$.

The term subnet will always refer to AA subnet unless otherwise qualified.

The following summarizes the dependencies of these definition:

$$\{ \text{ Willard } \} \subset \{ \text{ Kelley } \} \subset \{ \text{ AA } \}$$

It is easy to verify the above containments. They follow immediately from the definitions. The following examples from [Sch97] show that the these containments are proper.

Example 1.3.2. The net - aka sequence - $(2, 1, 4, 3, 6, 5, ...)$ is a Kelley subnet of $(1, 2, 3, 4, 5, 6, ...)$. However, it is not a Willard subnet.

Example 1.3.3. The net - aka sequence - $(0, 5, 6, 7, 8, ...)$ is an AA subnet of $(1, 5, 6, 7, 8, ...)$. However, it is not a Kelley subnet.

One should also note that none of the above notions of subnet are equivalent to subsequences when the directed set is $\mathbb{N}$. For instance $(1, 1, 2, 2, 2, ...)$ is a Willard subnet of $(1, 2, 2, 2, 2, ...)$ but not a subsequence in the traditional sense.

Theorem 1.3.4. *Fix a set X. If α, β are nets in X and $\mathcal{F}, \mathcal{G}$ are filters on X, then $\mathcal{G} \supseteq \mathcal{F}$ if and only if $\eta(\mathcal{G})$ is a subnet of $\eta(\mathcal{F})$.*

Proof. Suppose $\mathcal{G} \supseteq \mathcal{F}$. We then have by Theorem 1.2.14

$$\mathcal{E}(\eta(\mathcal{G})) = \mathcal{G} \supseteq \mathcal{F} = \mathcal{E}(\eta(\mathcal{F}))$$

so that $\eta(\mathcal{G})$ is a subnet of $\eta(\mathcal{F})$ by definition.

If $\eta(\mathcal{G})$ is a subnet of $\eta(\mathcal{F})$, then again by Theorem 1.2.14 and definition of subnet

$$\mathcal{G} = \mathcal{E}(\eta(\mathcal{G})) \supseteq \mathcal{E}(\eta(\mathcal{F})) = \mathcal{F}$$

as desired. QED

When $\mathcal{F}, \mathcal{G} \in \Phi(X)$ and $\mathcal{G} \supseteq \mathcal{F}$ one says that $\mathcal{G}$ *extends* $\mathcal{F}$.

The above result morally states that subnets and filter extensions are equivalent and is a key reason why AA subnets are taken here as the correct notion of subnet. However, those who prefer Willard subnets may be consoled by the corollary to the following result.

Lemma 1.3.5. *Suppose X is a set and α, β, and γ are nets. The following are equivalent.*

1. *$F \cap G \cap H \neq \emptyset$ for all $F \in \mathcal{E}(\alpha)$ and $G \in \mathcal{E}(\beta)$ and $H \in \mathcal{E}(\gamma)$.*
2. *$\mathcal{M} = \{M \supseteq F \cap G \cap H : F \in \mathcal{E}(\alpha) \ \wedge \ G \in \mathcal{E}(\beta) \ \wedge \ H \in \mathcal{E}(\gamma)\}$ is a filter on X.*
3. *There exists a filter $\mathcal{G} \supseteq \mathcal{E}(\alpha), \mathcal{E}(\beta), \mathcal{E}(\gamma)$.*
4. *There exists a net η which is a subnet of α, β, γ.*
5. *There exists a Willard subnet ω of α, β, γ such that whenever ζ is an AA subnet of α, β, γ with ω an AA subnet of ζ, we have $\omega \sim \zeta$.*

Proof. (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (1) is clear. Further, (5) $\implies$ (4) is immediate. It is left to show that (1) - (4) imply (5).

Assume (1) - (4). For each $a \in A$ and $b \in B$ and $c \in C$, define

$$T_{a,b,c} = \{\alpha(a') : a' \geq a\} \cap \{\beta(b') : b' \geq b\} \cap \{\gamma(c') : c' \geq c\}.$$

Condition (1) tells us each $T_{a,b,c}$ is non-empty. Let

$$\Omega = \{(a,b,c) \in A \times B \times C : \alpha(a) = \beta(b) = \gamma(c)\}$$

be a directed set with the product order. The fact that each tail $T_{a,b,c}$ is non-empty implies that Ω is cofinal in $A \times B \times C$; that is, for each $t \in A \times B \times C$ there is $s \in \Omega$ with $t \leq s$. It follows that the projections from Ω to A, B, C are final. Certainly, they are monotone. We define $\omega : \Omega \to X$ by $\omega(a,b,c) = \alpha(a) = \beta(b) = \gamma(c)$. It is then clear that ω is a Willard subnet of α, β, γ.

We next claim that $\mathcal{E}(\omega) = \mathcal{M}$. It is clear that $\mathcal{E}(\omega) \supseteq \mathcal{M}$ since $\mathcal{M}$ is the minimal filter extending $\mathcal{E}(\alpha), \mathcal{E}(\beta), \mathcal{E}(\gamma)$. Further, suppose $\omega \in_{\text{ev}} U \subseteq X$. We have some $(a,b,c) \in \Omega$ so that for all $(a',b',c') \geq (a,b,c)$ we have $\omega(a',b',c') \in U$. But this is $U \supseteq T_{a,b,c}$. Since $T_{a,b,c} \in \mathcal{M}$, we have $U \in \mathcal{M}$. Thus, $\mathcal{M} = \mathcal{E}(\omega)$.

Further, suppose ζ is a subnet of α, β, γ and ω is a subnet of ζ. Then $\mathcal{E}(\zeta) \supseteq \mathcal{E}(\alpha), \mathcal{E}(\gamma), \mathcal{E}(\beta)$. But $\mathcal{M} = \mathcal{E}(\omega)$ is the minimal filter with this property, so $\mathcal{E}(\zeta) \supseteq \mathcal{E}(\omega)$ and ζ is a subnet of ω. Thus, $\omega \sim \zeta$. QED

Remark 1.3.6. Observe that the above lemma can be adapted to any finite number of nets.

Corollary 1.3.7. *Suppose α is a net in a set X and β is a subnet of α. There exists a Willard subnet β' of α which is equivalent to β.*

Proof. We apply the previous lemma to α and β. Clearly α, β has subnet β. There is thus a Willard subnet β' of α, β so that whenever ζ is another subnet of α, β with β' a subnet of ζ, we have $\beta \sim \zeta$. Let $\zeta = \beta$. We have that $\beta' \sim \beta$. QED

Thus, whenever a subnet is produced, one may (up to equivalence) choose that subnet to be Willard.

Subnets and filter extensions interact with convergence in topological spaces exactly as desired.

Proposition 1.3.8. *Suppose X is a topological space with $x \in X$ and that α and $\mathcal{F}$ are respectively a net and filter in X converging to x.*

1. *If $\mathcal{G} \in \Phi(X)$ extends $\mathcal{F}$, then $\mathcal{G} \to x$.*
2. *If β is a subnet of α, then $\beta \to x$.*

Proof. (1) By Proposition 1.2.3, we have since $\mathcal{F} \to x$ that $\mathcal{F} \supseteq \mathcal{N}_x$. Since $\mathcal{G} \supseteq \mathcal{F}$, we have that $\mathcal{G} \supseteq \mathcal{N}_x$, and again by Proposition 1.2.3 that $\mathcal{G} \to x$.

(2) Since $\alpha \to x$, we have that $\mathcal{E}(\alpha) \to x$. By definition of subnet, we have that $\mathcal{E}(\beta) \supseteq \mathcal{E}(\alpha)$. By (1), we have that $\mathcal{E}(\beta) \to x$. Thus, $\eta(\mathcal{E}(\beta)) \to x$. Since $\eta(\mathcal{E}(\beta)) \sim \beta$, we have by Proposition 1.2.12 that $\beta \to x$. QED

Notation 1.3.9. As with sequences, it is often the case that when a net is not eventually within a set, it is desirable to take a subnet always outside of the set. The method for doing this is clear: simply restrict the net to a properly chosen cofinal subset of the domain. We end this section with a filter analogue of this process. Suppose $\mathcal{F}$ is a filter on a set X and $A \subseteq X$ is such that $A \cap F \neq \emptyset$ for each $F \in \mathcal{F}$. We may then construct the filter on X

$$\mathcal{F} \cap A := [\{F \cap A : F \in \mathcal{F}\}]$$

so that $\mathcal{F} \cap A \supseteq \mathcal{F}$ and $A \in \mathcal{F} \cap A$. Sometimes, we also wish to consider this as a filter on A. Thus, we define $\mathcal{F}|_A \in \Phi(A)$ by

$$\mathcal{F}|_A = \{F \cap A \subseteq A : F \in \mathcal{F}\}.$$

As subsets of X, the filter $\mathcal{F}|_A$ is a filter base for $\mathcal{F} \cap A$ in X. Thus, we write $[\mathcal{F}|_A] = \mathcal{F} \cap A$. If the set is not clear, we may also write $[\mathcal{F}|_A]_X$ for $\mathcal{F} \cap A$.

1.4 Ultrafilters and Universal Nets

Lemma 1.4.1. *Fix a set X and a filter $\mathcal{U}$ on X. The following are equivalent:*

1. *$\mathcal{U}$ is maximal under containment, that is for all $\mathcal{F} \in \Phi(X)$ if $\mathcal{F} \supseteq \mathcal{U}$ then $\mathcal{F} = \mathcal{U}$;*

2. *For all $A \subseteq X$ either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.*

Proof. Suppose $\mathcal{U}$ is maximal under containment. Suppose $A \subseteq X$. Consider

$$\mathcal{B} = \{A \cap U : U \in \mathcal{U}\}.$$

If $\mathcal{B} \ni \emptyset$ then there is $U \in \mathcal{U}$ with $U \subseteq X \setminus A$. Thus, $X \setminus A \in \mathcal{U}$ by definition of filter. Otherwise, we may have a filter $[\mathcal{B}] \in \Phi(X)$. If $U \in \mathcal{U}$, then $A \cap U \in \mathcal{B}$ and $A \cap U \subseteq U$ so that $U \in [\mathcal{B}]$. Therefore we have $[\mathcal{B}] \supseteq \mathcal{U}$ and $[\mathcal{B}] = \mathcal{U}$ by maximality. But then $X \in \mathcal{U}$ and $A = A \cap X \in \mathcal{U}$. So, as desired, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.

Now, suppose that for all $A \subseteq X$ either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$. Suppose $\mathcal{F} \in \Phi(X)$ and $\mathcal{F} \supseteq \mathcal{U}$. Suppose $F \in \mathcal{F}$. Since $X \setminus F \notin \mathcal{U}$, as otherwise $X \setminus F \in \mathcal{F}$ and $\emptyset = F \cap (X \setminus F) \in \mathcal{F}$ which is impossible, we have $F \in \mathcal{U}$. Therefore, $\mathcal{U} \supseteq \mathcal{F}$ and $\mathcal{U} = \mathcal{F}$. QED

Definition 1.4.2. Any filter satisfying either of the equivalent conditions of Lemma 1.4.1 is called an *ultrafilter*.

Corollary 1.4.3. *If $\mathcal{U}$ is an ultrafilter on a set X and*

$$U_1 \cup U_2 \in \mathcal{U}$$

then $U_1 \in \mathcal{U}$ or $U_2 \in \mathcal{U}$.

Proof. If $U_1 \notin \mathcal{U}$, then $X \setminus U_1 \in \mathcal{U}$. We then have $U_2 \supseteq (U_1 \cup U_2) \cap (X \setminus U_1) \in \mathcal{U}$ so that $U_2 \in \mathcal{U}$ as desired. QED

Unsurprisingly, there is a parallel notion for nets.

Definition 1.4.4. A net is called *universal* when it is equivalent to each of its subnets.

Theorem 1.4.5. *Suppose X is a set, ω is a net in X, and $\mathcal{U}$ a filter on X.*

1. *ω is universal if and only if $\mathcal{E}(\omega)$ is an ultrafilter.*
2. *$\mathcal{U}$ is an ultrafilter if and only if $\eta(\mathcal{U})$ is universal.*

Proof. (1) Suppose ω is universal. Let $\mathcal{F} \supseteq \mathcal{E}(\omega)$. We then have by Theorem 1.3.4 that $\eta(\mathcal{F})$ is a subnet of ω since ω is universal. This means that

$$\mathcal{E}(\omega) = \mathcal{E}(\eta(\mathcal{F})) = \mathcal{F}$$

and $\mathcal{E}(\omega)$ is an ultrafilter.

Next, suppose $\mathcal{E}(w)$ is an ultrafilter. Suppose α is a subnet of ω. Then we have $\mathcal{E}(\alpha) \supseteq \mathcal{E}(\omega)$ by Theorem 1.3.4 and $\mathcal{E}(\alpha) = \mathcal{E}(\omega)$ by definition of ultrafilter. We then have that $\alpha \sim \omega$ and ω is universal by definition.

(2) Suppose $\mathcal{U}$ is an ultrafilter. Then since $\mathcal{U} = \mathcal{E}(\eta(\mathcal{U}))$ we have $\eta(\mathcal{U})$ is an ultrafilter by (1). On the other hand, if $\eta(\mathcal{U})$ is universal, then by (1) we have that $\mathcal{U} = \mathcal{E}(\eta(\mathcal{U}))$ is universal. QED

An entirely reasonable question is whether ultrafilters/universal nets exist.

Example 1.4.6. If X is a set and $x \in X$, then $[x]$ is an ultrafilter on X.

However, for less trivial examples one must rely on the following result.

Theorem 1.4.7 (Ultrafilter Lemma)**.** *If $\mathcal{F}$ is a filter on a set X, there exists an ultrafilter $\mathcal{U}$ on X with $\mathcal{U} \supseteq \mathcal{F}$.*

Proof. Define

$$\mathcal{P} = \{\mathcal{G} \in \Phi(X) : \mathcal{G} \supseteq \mathcal{F}\}$$

which is non-empty since $\mathcal{F} \in \mathcal{P}$. Consider $\mathcal{P}$ as a poset under set containment. Let $\mathcal{C} \subseteq \mathcal{P}$ be a chain. It is easy to check that $\bigcup \mathcal{C}$ is an upper bound for $\mathcal{C}$ in $\mathcal{P}$. By Zorn's Lemma, we have that $\mathcal{P}$ has a maximal element. This is exactly an ultrafilter extension of $\mathcal{F}$. QED

Corollary 1.4.8. *Every net has a universal subnet.*

Corollary 1.4.9. *There exist ultra filters which are not point filters.*

Proof. Let X be any infinite set and $\mathcal{F}$ be the Frechét filter on X,

$$\mathcal{F} = \{F \subseteq X : X \setminus F \text{ is finite}\}.$$

By the Ultrafilter Lemma, there is an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$. Since for each $x \in X$ one has $X \setminus \{x\} \in \mathcal{F}$ so that $\{x\}$ cannot be in $\mathcal{U}$. QED

As it turns out, the Ultrafilter Lemma, while strictly weaker than the Axiom of Choice, is not provable from ZF alone, nor ZF along with Dependent Choice.[1] As a consequence, it is not possible to produce explicit examples of ultrafilters which are not generated by a singe point.

1.5 Functions, Nets, and Filters

Notation 1.5.1. If α is a net in X and $f : X \to Y$ is a mapping of sets, we write $f(\alpha)$ for the net $f \circ \alpha$ in Y.

Proposition 1.5.2. *If $\mathcal{F}$ is a filter on X and $f : X \to Y$ is a function, then the* image filter

$$f(\mathcal{F}) = [\{f(F) \subseteq Y : F \in \mathcal{F}\}]$$

is a filter on Y.

Proof. By Proposition 1.2.8, we need only check for each $F_1, ..., F_n \in \mathcal{F}$ that

$$\bigcap_{i=1}^{n} f(F_i) \neq \emptyset.$$

Indeed, this follows as

$$\bigcap_{i=1}^{n} f(F_i) \supseteq f\left(\bigcap_{i=1}^{n} F_n\right).$$

and $\bigcap_{i=1}^{n} F_n \neq \emptyset$ since $\mathcal{F}$ is a filter. QED

Corollary 1.5.3. *If $\mathcal{F}$ is a filter on X and $f : X \to Y$ is a function, then $G \in f(\mathcal{F})$ if and only if there is $F \in \mathcal{F}$ with $G \supseteq f(F)$.*

Proposition 1.5.4. *Let X be a set. If α is a net in X and $\mathcal{F}$ a filter on X, then*

1. $f(\mathcal{E}(\alpha)) = \mathcal{E}(f(\alpha))$;
2. $f(\eta(\mathcal{F})) \sim \eta(f(\mathcal{F}))$.

[1] See the discussion in chapter 14 of [Sch97] as well as [PS77]

Proof. (1) Suppose $G \in f(\mathcal{E}(\alpha))$. There is then $F \in \mathcal{E}(\alpha)$ with $G \supseteq f(F)$. Since $\alpha \in_{\text{ev}} F$, we have $f(\alpha) \in_{\text{ev}} G$ and so $G \in \mathcal{E}(f(\alpha))$. Likewise, if $G \in \mathcal{E}(f(\alpha))$, then $f(\alpha) \in_{\text{ev}} G$. Then $\alpha \in_{\text{ev}} f^{-1}(G)$. It follows that $f^{-1}(G) \in \mathcal{E}(\alpha)$ so $G \supseteq f(f^{-1}(G)) \in f(\mathcal{E}(\alpha))$ and $G \in f(\mathcal{E}(\alpha))$. Thus, $f(\mathcal{E}(\alpha)) = \mathcal{E}(f(\alpha))$ as desired.

(2) Observe that

$$\begin{aligned}
\eta(f(\mathcal{F})) &= \eta(f(\mathcal{E}(\eta(\mathcal{F})))) && \text{(by Theorem 1.2.14)} \\
&= \eta(\mathcal{E}(f(\eta(\mathcal{F})))) && \text{(by (1))} \\
&\sim f(\eta(\mathcal{F})) && \text{(by Theorem 1.2.14)}
\end{aligned}$$

as desired. QED

Corollary 1.5.5. *Suppose α and β are nets in a set X and $f : X \to Y$. If $\alpha \sim \beta$ then $f(\alpha) \sim f(\beta)$.*

Proof. This follows as

$$\mathcal{E}(f(\alpha)) = f(\mathcal{E}(\alpha)) = f(\mathcal{E}(\beta)) = \mathcal{E}(f(\beta)).$$

QED

Corollary 1.5.6. *If $f : X \to Y$ and $g : Y \to Z$ are set mappings and $\mathcal{F}$ is a filter on X, then $g(f(\mathcal{F})) = (g \circ f)(\mathcal{F})$.*

Proof. We observe that

$$\begin{aligned}
(g \circ f)(\mathcal{F}) &= (g \circ f)(\mathcal{E}(\eta(\mathcal{F}))) \\
&= \mathcal{E}((g \circ f)(\eta(\mathcal{F}))) \\
&= \mathcal{E}(g(f(\eta(\mathcal{F})))) \\
&= g(\mathcal{E}(f(\eta(\mathcal{F})))) \\
&= g(f(\mathcal{E}(\eta(\mathcal{F})))) \\
&= g(f(\mathcal{F}))
\end{aligned}$$

as desired. QED

Proposition 1.5.7. *If $\mathcal{U}$ is an ultrafilter on a set X and $f : X \to Y$ is a set mapping, then $f(\mathcal{U})$ is an ultrafilter.*

Proof. Suppose $A \subseteq Y$. Either $f^{-1}(A) \in \mathcal{U}$ or $X \setminus f^{-1}(A) \in \mathcal{U}$. If $f^{-1}(A) \in \mathcal{U}$, then $A \supseteq f(f^{-1}(A)) \in f(\mathcal{U})$ so $A \in f(\mathcal{U})$. Otherwise, an identical argument shows that $Y \setminus A \in f(\mathcal{U})$. QED

Corollary 1.5.8. *If ω is a universal net in a set X and $f : X \to Y$ is a set mapping, then $f(\omega)$ is universal.*

Proposition 1.5.9. *If $f : X \to Y$ is a set mapping, $\mathcal{F}$ a filter on X and $\mathcal{G}$ a filter on Y with $\mathcal{G} \supseteq f(\mathcal{F})$, then*

$$f^{-1}(\mathcal{G}) = [\{f^{-1}(G) : G \in \mathcal{G}\}]$$

is a filter and $f(f^{-1}(\mathcal{G})) \supseteq \mathcal{G}$.

Proof. We first show that $f^{-1}(\mathcal{U})$ is a filter. It suffices to show that if $U \in \mathcal{U}$ then $f^{-1}(U) \neq \emptyset$. Let $U \in \mathcal{U}$. Since $X \in \mathcal{F}$, we have $f(X) \in f(\mathcal{F})$ so that $f(X) \in \mathcal{U}$. Therefore, $U \cap f(X) \neq \emptyset$ and $f^{-1}(U) \neq \emptyset$.

We now show the desired containment. Let $G \in \mathcal{G}$ and $f^{-1}(G) \in f^{-1}(\mathcal{G})$. Since $f(f^{-1}(\mathcal{G})) \ni f(f^{-1}(G)) \subseteq G$, we have $G \in f(f^{-1}(\mathcal{G}))$ as desired. QED

1.6 Miscellaneous Results on Nets and Filters

This section contains an array of results on filters and nets which will be helpful in future chapters.

Proposition 1.6.1. *If $\mathcal{F}, \mathcal{G}$ are filters on a set X, then $\mathcal{F} \cap \mathcal{G}$ is a filter on X.*

Proof. $X \in \mathcal{F}, \mathcal{G}$ so $X \in \mathcal{F} \cap \mathcal{G}$. Since $\emptyset \notin \mathcal{F}, \mathcal{G}$ we may be assured that $\emptyset \notin \mathcal{F} \cap \mathcal{G}$.

Suppose $F \in \mathcal{F} \cap \mathcal{G}$ and $X \supseteq G \supseteq F$. Then $G \in \mathcal{F}$ and $G \in \mathcal{G}$, so $G \in \mathcal{F} \cap \mathcal{G}$.

Suppose $G, F \in \mathcal{F} \cap \mathcal{G}$. Then $F \cap G \in \mathcal{F}, \mathcal{G}$, so $F \cap G \in \mathcal{F} \cap \mathcal{G}$. By definition, $\mathcal{F} \cap \mathcal{G}$ is a filter on X. QED

Corollary 1.6.2. *Arbitrary intersections of filters are filters.*

Proposition 1.6.3. *If X is a topological space with $x \in X$ with filters $\mathcal{F}, \mathcal{G} \to x$, then $\mathcal{F} \cap \mathcal{G} \to x$.*

Proof. Since $\mathcal{F}, \mathcal{G} \to x$, we have that $\mathcal{F}, \mathcal{G} \supseteq \mathcal{N}_x$. Then of course $\mathcal{F} \cap \mathcal{G} \supseteq \mathcal{N}_x$. Thus, $\mathcal{F} \cap \mathcal{G} \to x$. QED

Proposition 1.6.4. *Suppose $\mathcal{F}, \mathcal{G}$ are filters on a set X. If an ultrafilter $\mathcal{U} \supseteq \mathcal{F} \cap \mathcal{G}$ then $\mathcal{U} \supseteq \mathcal{F}$ or $\mathcal{U} \supseteq \mathcal{G}$.*

Proof. Let $\mathcal{U} \supseteq \mathcal{F} \cap \mathcal{G}$ be an ultrafilter. Suppose $\mathcal{U} \not\supseteq \mathcal{F}, \mathcal{G}$. We then find $F \in \mathcal{F} \setminus \mathcal{U}$ and $G \in \mathcal{G} \setminus \mathcal{U}$. It follows that $F \cup G \in \mathcal{F} \cap \mathcal{G}$ and $F \cup G \in \mathcal{U}$. Further, $X \setminus F, X \setminus G \in \mathcal{U}$ since $\mathcal{U}$ is an ultrafilter. But then $(X \setminus F) \cap (X \setminus G) \in \mathcal{U}$ and

$$X \setminus \big((X \setminus F) \cap (X \setminus G)\big) = F \cup G \notin \mathcal{U}.$$

This is a contradiction. Thus, $\mathcal{U} \supseteq \mathcal{F}$ or $\mathcal{U} \supseteq \mathcal{G}$ as desired. QED

Definition 1.6.5. If $\{\alpha_s : s \in S\}$ is a collection of nets in a set X, define

$$\bigwedge_{s\in S} \alpha_s = \eta(\bigcap_{s\in S} \mathcal{E}(\alpha_s))$$

Call this net the *intertwining* of the α_s. If $\{\alpha_s : s \in S\} = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ write $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n$ for $\bigwedge_{s\in S} \alpha_s$.

The following results on the intertwining of nets are formulated for two nets, but extend to arbitrarily many nets.

Lemma 1.6.6. *If α, β are nets in a set X, then $\mathcal{E}(\alpha \wedge \beta) = \mathcal{E}(\alpha) \cap \mathcal{E}(\beta)$.*

Proof. This follows from the definition of intertwining and Theorem 1.2.14. QED

This intertwining of nets satisfies the following universal property.

Proposition 1.6.7. *If α and β are nets in a set X, then α and β are subnets of $\alpha \wedge \beta$ and for any net ζ having α and β as subnets, $\alpha \wedge \beta$ is a subnet of ζ.*

Proof. Since

$$\mathcal{E}(\alpha), \mathcal{E}(\beta) \supseteq \mathcal{E}(\alpha) \cap \mathcal{E}(\beta) = \mathcal{E}(\alpha \wedge \beta)$$

we have that α and β are subnets of $\alpha \wedge \beta$. If ζ is a net in X which contains α and β as subnets, then we have $\mathcal{E}(\alpha), \mathcal{E}(\beta) \supseteq \mathcal{E}(\zeta)$ so that $\mathcal{E}(\alpha) \cap \mathcal{E}(\beta) \supseteq \mathcal{E}(\zeta)$ and $\alpha \wedge \beta$ is a subnet of ζ. QED

Example 1.6.8. If $\alpha, \beta : \mathbb{N} \to X$ are sequences and we define $\gamma : \mathbb{N} \to X$ by

$$\gamma(n) = \begin{cases} \alpha(k) & n = 2k \\ \beta(k) & n = 2k + 1 \end{cases}$$

then $\gamma \sim \alpha \wedge \beta$.

Proposition 1.6.9. *If $\mathcal{F}, \mathcal{G}$ are filters on a set X and $f : X \to Y$ is a mapping of sets, then $f(\mathcal{F} \cap \mathcal{G}) = f(\mathcal{F}) \cap f(\mathcal{G})$.*

Proof. Suppose $H \in f(\mathcal{F} \cap \mathcal{G})$. There is then $H' \in \mathcal{F} \cap \mathcal{G}$ so that $H \supseteq f(H')$. Since $H' \in \mathcal{F}$, we have $f(H') \in f(\mathcal{F})$. Likewise, $f(H') \in f(\mathcal{G})$. Therefore, $f(H') \in f(\mathcal{F}) \cap f(\mathcal{G})$ and $H \in \mathcal{F} \cap \mathcal{G}$. We then have $f(\mathcal{F} \cap \mathcal{G}) \subseteq f(\mathcal{F}) \cap f(\mathcal{G})$.

On the other hand, suppose $H \in f(\mathcal{F}) \cap f(\mathcal{G})$. We may find $F \in \mathcal{F}$ and $G \in \mathcal{G}$ so that $H \supseteq f(F), f(G)$. We the have that $F \cup G \in \mathcal{F} \cap \mathcal{G}$ and $H \supseteq f(F \cup H)$. Therefore, $H \in \mathcal{F} \cap \mathcal{G}$ and $f(\mathcal{F} \cap \mathcal{G}) \supseteq f(\mathcal{F}) \cap f(\mathcal{G})$ so that $f(\mathcal{F} \cap \mathcal{G}) = f(\mathcal{F}) \cap f(\mathcal{G})$ as desired. QED

Corollary 1.6.10. *If α and β are nets in a set X and $f : X \to Y$ is a mapping of sets, then $f(\alpha \wedge \beta) \sim f(\alpha) \wedge f(\beta)$.*

Proof. We compute

$$\begin{aligned} f(\alpha \wedge \beta) &= f(\eta(\mathcal{E}(\alpha) \cap \mathcal{E}(\beta))) \\ &\sim \eta(f(\mathcal{E}(\alpha) \cap \mathcal{E}(\beta))) \\ &= \eta(f(\mathcal{E}(\alpha)) \cap f(\mathcal{E}(\beta))) \\ &= \eta(\mathcal{E}(f(\alpha)) \cap \mathcal{E}(f(\beta))) \\ &= f(\alpha) \wedge f(\beta), \end{aligned}$$

which is the desired result. QED

Proposition 1.6.11. *If X and Y are sets and $\mathcal{F}, \mathcal{G}$ are filters on X and Y respectively, then*

$$\mathcal{F} \times \mathcal{G} = [\{F \times G : F \in \mathcal{F} \text{ and } G \in \mathcal{G}\}]$$

is a filter on $X \times Y$. This is called the product filter *of $\mathcal{F}$ and $\mathcal{G}$.*

Proof. It is clear that the conditions of Proposition 1.2.8 are satisfied. QED

Corollary 1.6.12. *If X and Y are sets and $\mathcal{F}, \mathcal{G}$ are filters on X and Y respectively, then $H \in \mathcal{F} \times \mathcal{G}$ if and only if there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ so that $H \supseteq F \times G$.*

Proof. This follows from the fact that if $A_1, A_2 \subseteq X$ and $B_1, B_2 \subseteq Y$ then

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$$

and the remark following Proposition 1.2.8. QED

Proposition 1.6.13. *If X and Y are sets with $\mathcal{F}, \mathcal{G}$ filters on X and Y respectively and $\pi : X \times Y \to X$ is the usual projection on the first coordinate, then $\pi(\mathcal{F} \times \mathcal{G}) = \mathcal{F}$.*

Proof. If $F \in \mathcal{F}$, then $F \times Y \in \mathcal{F} \times \mathcal{G}$ and $F = \pi(F \times Y) \in \pi(\mathcal{F} \times \mathcal{G})$. Likewise, if $H \in \pi(\mathcal{F} \times \mathcal{G})$ then there are some $F \in \mathcal{F}$ and $G \in \mathcal{G}$ so that $H \supseteq \pi(F \times G)$. Then $H \supseteq F$ and $H \in \mathcal{F}$. We conclude that $\pi(\mathcal{F} \times \mathcal{G}) = \mathcal{F}$. QED

Proposition 1.6.14. *Fix sets X and Y. If $\mathcal{H}$ is a filter on $X \times Y$, there exist filters $\mathcal{F}$ on X and $\mathcal{G}$ on Y so that $\mathcal{H} \supseteq \mathcal{F} \times \mathcal{G}$.*

Proof. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the usual projections. The filters $\mathcal{F} = \pi_1(\mathcal{H})$ and $\mathcal{G} = \pi_2(\mathcal{H})$ have the desired properties. QED

Products of filters interact well with products of functions and diagonals.

Proposition 1.6.15. *Fix sets X, Y, W and Z, filters $\mathcal{F} \in \Phi(X)$ and $\mathcal{G} \in \Phi(Y)$, and functions $f : X \to W$ and $g : Y \to Z$. When $f \times g : X \times Y \to W \times Z$ is given by $(f \times g)(x, y) = (f(x), g(y))$, one has $(f \times g)(\mathcal{F} \times \mathcal{G}) = f(\mathcal{F}) \times g(\mathcal{G})$.*

Proof. Let $H \in (f \times g)(\mathcal{F} \times \mathcal{G})$. There is then $H' \in \mathcal{F} \times \mathcal{G}$ so that $H \supseteq (f \times g)(H')$. Since $H' \in \mathcal{F} \times \mathcal{G}$, there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ so that $H' \supseteq F \times G$. Thus,

$$H \supseteq (f \times g)(F \times G) = f(F) \times g(G) \in f(\mathcal{F}) \times g(\mathcal{G})$$

and $H \inf(\mathcal{F}) \times g(\mathcal{G})$. We now have $(f \times g)(\mathcal{F} \times \mathcal{G}) \subseteq f(\mathcal{F}) \times g(\mathcal{G})$.

For the other direction, suppose $H \in f(\mathcal{F}) \times g(\mathcal{G})$. There are then $H_1 \in f(\mathcal{F})$ and $H_2 \in g(\mathcal{G})$ so that $H \supseteq H_1 \times H_2$. Since $H_1 \in f(\mathcal{F})$, there is some $F \in \mathcal{F}$ so that $H_1 \supseteq f(F)$. Likewise, there is some $G \in \mathcal{G}$ so that $H_2 \supseteq g(G)$. We then have that

$$H \supseteq f(F) \times g(G) = (f \times g)(F \times G) \in (f \times g)(\mathcal{F} \times \mathcal{G}) = f(\mathcal{F})$$

and $H \in (f \times g)(\mathcal{F} \times \mathcal{G}) = f(\mathcal{F})$.

We conclude $(f \times g)(\mathcal{F} \times \mathcal{G}) = f(\mathcal{F}) \times g(\mathcal{G})$ as desired. QED

Proposition 1.6.16. *If X is a set, define the* diagonal map $\Delta : X \to X \times X$ *to be given by $x \mapsto (x, x)$. If $\mathcal{F} \in \Phi(X)$, then $\Delta(\mathcal{F}) \supseteq \mathcal{F} \times \mathcal{F}$.*

Proof. By Proposition 1.6.14, we have that

$$\begin{aligned}\Delta(\mathcal{F}) &\supseteq \pi_1(\Delta(\mathcal{F})) \times \pi_2(\Delta(\mathcal{F})) \\ &= (\pi_1 \circ \Delta)(\mathcal{F}) \times (\pi_2 \circ \Delta)(\mathcal{F}) \\ &= \mathcal{F} \times \mathcal{F}.\end{aligned}$$

For the other containment, suppose $H \in \mathcal{F} \times F$. There are then $F_1, F_2 \in \mathcal{F}$ so that $H \supseteq F_1 \times F_2$. We then see that $H \supseteq \Delta(F_1 \cap F_2)$ so that $H \in \Delta(\mathcal{F})$. We conclude $\Delta(\mathcal{F}) = \mathcal{F} \times \mathcal{F}$. QED

Definition 1.6.17. Suppose X and Y are sets and α, β are nets in X and Y respectively. Define the *product net* in $X \times Y$ by

$$(\alpha, \beta) : \mathrm{dom}(\alpha) \times \mathrm{dom}(\beta) \to X \times Y$$

by $(\alpha, \beta)(i, k) = (\alpha_i, \beta_k)$ for each $i \in \mathrm{dom}(\alpha)$ and $k \in \mathrm{dom}(\beta)$.

Proposition 1.6.18. *Let X and Y be sets with $\alpha, \mathcal{F}$ respectively a net and filter on X and $\beta, \mathcal{G}$ respectively a net and filter on Y.*

1. $\mathcal{E}((\alpha, \beta)) = \mathcal{E}(\alpha) \times \mathcal{E}(\beta)$.
2. $\eta(\mathcal{F} \times \mathcal{G}) \sim (\eta(\mathcal{F}), \eta(\mathcal{G}))$.

Proof. (1) Suppose (α, β) is eventually in some $U \subseteq X \times Y$. We then have some $(a_0, b_0) \in A \times B$ so that the tail

$$T = \{(\alpha(a), \beta(b)) : (a, b) \geq (a_0, b_0)\} \subseteq U.$$

Let $T_A = \{\alpha(a) : a \geq a_0\}$ and $T_B = \{\beta(b) : b \geq b_0\}$. We see that $T = T_A \times T_b$. Certainly, $T_A \in \mathcal{E}(\alpha)$ and $T_B \in \mathcal{E}(\beta)$. Therefore, since $U \in \mathcal{E}(\alpha) \times \mathcal{E}(\beta)$.

Now, assume that $U \in \mathcal{E}(\alpha) \times \mathcal{E}(\beta)$. We have some $F_1 \in \mathcal{E}(\alpha)$ and $F_2 \in \mathcal{E}(\beta)$ so that $U \supseteq F_1 \times F_2$. But then α, β are eventually in F_1 and F_2 respectively, so that (α, β) is eventually in $F_1 \times F_2$. We conclude that (α, β) is eventually in U. This concludes (1) by double containment.

(2) Follows directly from (1). Observe

$$\begin{aligned} \mathcal{E}(\eta(\mathcal{F} \times \mathcal{G})) &= \mathcal{F} \times \mathcal{G} \\ &= \mathcal{E}(\eta(\mathcal{F})) \times \mathcal{E}(\alpha_{\mathcal{G}}) \\ &= \mathcal{E}((\eta(\mathcal{F}), \eta(\mathcal{G}))). \end{aligned}$$

This is exactly that $\eta(\mathcal{F} \times \mathcal{G}) \sim (\eta(\mathcal{F}), \eta(\mathcal{G}))$ as desired. QED

Corollary 1.6.19. *If X and Y are sets with α, β are nets in X and Y respectively and $\pi : X \times Y \to X$ is the usual projection, then $\pi(\alpha, \beta) \sim \alpha$.*

Proof. We compute that

$$\begin{aligned} \mathcal{E}(\pi(\alpha, \beta)) &= \pi(\mathcal{E}((\alpha, \beta))) \\ &= \pi(\mathcal{E}(\alpha) \times \mathcal{E}(\beta)) \\ &= \mathcal{E}(\alpha) \end{aligned}$$

which is exactly that $\pi(\alpha, \beta) \sim \alpha$. QED

Chapter 2

Convergence Spaces

With the last chapter's preliminaries in place, we may now use nets and filters to generalize topological spaces with convergence spaces. In the first section we will present the definition and basic properties of such spaces and continuous maps between them. In the second, we will explore how convergence spaces relate to topological spaces. In the third, we will introduce methods for constructing convergence spaces. The fourth and fifth sections will extend the topological notions of separation and compactness to convergence spaces. In the sixth section, we will discuss several significant classes of convergence spaces. The exposition here mainly follows that in [Pat14] and - particularly in 2.6 - [BB02].

2.1 Convergence Structures

In this section, we will define a type of space which uses the convergence of filters as its primitive notion and define from this a notion of convergence of nets. We will then prove some basic results on this convergence. Further, we will show that not only does this convergence include the convergence of nets and filters in topological spaces but also the almost everywhere convergence from Section 1.1. Following this, we will define continuous functions between convergence spaces and introduce the category of convergence spaces.

Definition 2.1.1. A *convergence structure* on a set X is a relation $\to \subseteq \Phi(X) \times X$ so that

1. $[x] \to x$ for each $x \in X$;
2. if $\mathcal{F}, \mathcal{G} \in \Phi(X)$ and $x \in X$ with $\mathcal{F} \to x$ and $\mathcal{G} \supseteq \mathcal{F}$ then $\mathcal{G} \to x$;
3. if $\mathcal{F}, \mathcal{G} \in \Phi(X)$ and $x \in X$ with $\mathcal{F} \to x$ and $\mathcal{G} \to x$ then $\mathcal{F} \cap \mathcal{G} \to x$.

The symbol $\to$ is read as *converges to*. A set together with a convergence structure is called a *convergence space*.

Example 2.1.2. By Propositions 1.2.18, 1.3.8, and 1.6.3 we have that the convergence of filters in a topological space as defined in Definition 1.2.16 gives a convergence structure. Given a topological space with underlying set X, we will denote X with this convergence structure by $\mathfrak{C}(X)$. Convergence spaces which can be obtained in this way are called *topological convergence spaces*.

The reasoning behind Definition 1.2.16 may now be inverted to define a convergence of nets in a convergence space which satisfies the net analogues to the properties set out in Definition 2.1.1.

Definition 2.1.3. If X is a convergence space, $x \in X$, and α a net in X, say that α *converges to* x and write $\alpha \to x$ when $\mathcal{E}(\alpha) \to x$.

Observe that if the convergence structure on X is inherited from a topological space, this is just the usual notion of convergence.

Proposition 2.1.4. *Suppose X is a convergence space with $x \in X$.*

1. *If α is a constant net at x, then $\alpha \to x$.*
2. *If α is a net in X converging to x and β is a subnet of α, then $\beta \to x$.*
3. *If α and β are nets in X both converging to x, then $\alpha \wedge \beta \to x$.*

Proof. (1) Suppose α is a constant net at x. By Proposition 1.2.17 we have that $\mathcal{E}(\alpha) = [x]$. Since $[x] \to x$ we have $\alpha \to x$.

(2) Suppose α is a net in X converging to x and β is a subnet of α. We have by definition that $\mathcal{E}(\beta) \supseteq \mathcal{E}(\alpha)$. Since $\alpha \to x$ we have $\mathcal{E}(\alpha) \to x$ and thus $\mathcal{E}(\beta) \to x$. Therefore, $\beta \to x$.

(3) Suppose α and β are nets in X both converging to x. We then have by Lemma 1.6.6 that $\mathcal{E}(\alpha \wedge \beta) = \mathcal{E}(\alpha) \cap \mathcal{E}(\beta)$. Further, since $\alpha, \beta \to x$ we have that $\mathcal{E}(\alpha), \mathcal{E}(\beta) \to x$. From this it follows that $\mathcal{E}(\alpha) \cap \mathcal{E}(\beta) \to x$ so that $\alpha \wedge \beta \to x$. QED

Remark 2.1.5. It is possible to define convergence structures directly via nets instead of filters. One way is to single out some specified set of nets as primitive, define a convergence structure on these satisfying the properties of Proposition 2.1.4, and then use equivalence of nets to extend this convergence to all nets. There are a variety of difficulties here, the first of which is how to choose a set of primitive nets large enough to "see" any other net via equivalence. The simplest solution is to choose the set of primitive nets to be the derived nets of filters. Of course, this hardly avoids using filters.

Another way around this problem is to sidestep the set theoretic concern by viewing the collection of nets as a category. Then, a net based convergence structure can be realized as a functor satisfying various properties. The details of this method are discussed in Appendix B.2.

We will now present some basic properties of net convergence.

Proposition 2.1.6. *Suppose X is a convergence space. If $\mathcal{F}$ is a filter on X converging to a point $x \in X$, then $\eta(\mathcal{F}) \to x$.*

Proof. This follows from the fact that $\mathcal{E}(\eta(\mathcal{F})) = \mathcal{F}$. QED

Proposition 2.1.7. *If X is a convergence space and $x \in X$ and α and β are nets in X with $\alpha \sim \beta$, then $\alpha \to x$ if and only if $\beta \to x$.*

Proof. This follows since $\mathcal{E}(\alpha) = \mathcal{E}(\beta)$. QED

Proposition 2.1.8. *Let X be a convergence space. Suppose $\alpha : I \to X$ is a net. For each $i \in I$, define*

$$I_{\geq i} = \{j \in I : j \geq i\}.$$

If there is $i \in I$ and $x \in X$ and cofinal $I_1, ..., I_n \subseteq I$ so that

1. $I_{\geq i} \subseteq \bigcup_{k=1}^{n} I_k$
2. $\alpha|_{I_k} \to x$ *for each* $k = 1, ..., n$

then $\alpha \to x$.

Proof. This follows from the observation that

$$\mathcal{E}(\alpha) = \bigcap_{k=1}^{n} \mathcal{E}(\alpha|_{I_k}).$$

QED

Corollary 2.1.9. *Suppose X is a convergence space and $\alpha : I \to X$ is a net. Fix $i \in I$. For all $x \in X$, we have $\alpha \to x$ if and only if $\alpha|_{I_{\geq i}} \to x$. This restriction of α is called a* tail*.*

We have seen that the convergence of nets and filters in topological spaces is given by a convergence structure. We will now show that almost everywhere convergence is as well.

Lemma 2.1.10. *Let X and Y be sets with Y a convergence space and $Z = \{f : X \to Y\}$. Suppose that Λ and Γ are nets in Z and that $x \in X$ and $y \in Y$. If $\Lambda \sim \Gamma$ and $\Lambda(x) \to y$ then $\Gamma(x) \to y$.*

Proof. Define $\mathrm{ev}_x : Z \to Y$ by $f \mapsto f(x)$. Since $\Lambda \sim \Gamma$, we have by Corollary 1.5.5 that

$$\Lambda(x) = \mathrm{ev}_x(\Lambda) \sim \mathrm{ev}_x(\Gamma) = \Gamma(x).$$

Since $\Lambda(x) \to y$ we then have $\Gamma(x) \to y$. QED

Proposition 2.1.11. *Fix a measure space (X, Σ, μ) and set*

$$\mathcal{M}(X) = \{f : X \to \mathbb{R} \mid f \text{ is measurable}\}.$$

Define the almost everywhere convergence structure *on $\mathcal{M}(X)$ by saying if $\mathscr{F}$ is a filter on $\mathcal{M}(X)$ and $f \in \mathcal{M}(X)$, then $\mathscr{F} \to f$ iff $\eta(\mathscr{F})$ converges to f almost everywhere. This is a convergence structure.*

Proof. First, suppose $f \in \mathcal{M}(X)$. Recall from Proposition 1.2.17 that $\eta([f])$ is equivalent to the constant net at f. By the previous lemma, we have that $\eta([f])$ converges to f almost everywhere so that $[f] \to f$.

Suppose $f \in \mathcal{M}(X)$ and $\mathcal{F}, \mathcal{G}$ are filters on $\mathcal{M}(X)$ with $\mathcal{G} \supseteq \mathcal{F}$ and $\mathcal{F} \to f$. We have that $\eta(\mathcal{G})$ is a subnet of $\eta(\mathcal{F})$. We then have for each $x \in X$ that $\mathrm{ev}_x(\eta(\mathcal{G}))$ is a subnet of $\mathrm{ev}_x(\eta(\mathcal{F}))$. From this it follows that if $\mathrm{ev}_x(\eta(\mathcal{F}))$ converges then so does $\mathrm{ev}_x(\eta(\mathcal{G}))$ and to the same limit. Therefore $\eta(\mathcal{G})$ converges to f almost everywhere and $\mathcal{G} \to f$.

Suppose $f \in \mathcal{M}(X)$ and $\mathcal{F}, \mathcal{G}$ are filters on $\mathcal{M}(X)$ with $\mathcal{F}, \mathcal{G} \to f$. We have some set $A, B \in \Sigma$ so that $\mu(X \setminus A) = \mu(X \setminus B) = 0$ and $\eta(\mathcal{F})(x) \to f(x)$ for all $x \in A$ and $\eta(\mathcal{G})(x) \to f(x)$ for all $x \in B$. We observe that

$$\begin{aligned}\mu(X \setminus (A \cap B)) &= \mu((X \setminus A) \cup (X \setminus B)) \\ &\leq \mu(X \setminus A) + \mu(X \setminus B) \\ &= 0\end{aligned}$$

and that if $x \in A \cap B$ then

$$\begin{aligned}\eta(\mathcal{F} \cap \mathcal{G})(x) &= (\eta(\mathcal{F}) \wedge \eta(\mathcal{G}))(x) \\ &= \eta(\mathcal{F})(x) \wedge \eta(\mathcal{G})(x) \\ &\to f(x).\end{aligned}$$

Thus, $\eta(\mathcal{F} \cap \mathcal{G})$ converges to f almost everywhere and $\mathcal{F} \cap \mathcal{G} \to f$.

We conclude that this relation is a convergence structure. QED

Corollary 2.1.12. *When (X, Σ, μ) is a measure space and $\mathcal{M}(X)$ is given the almost everywhere convergence structure, a net Λ in $\mathcal{M}(X)$ converges to $f \in \mathcal{M}(X)$ if and only if it converges almost everywhere.*

While we have seen that convergence spaces describe a large collection of convergence notions, including both topological convergence and almost everywhere convergence, there are still useful notions of convergence which are not given by any convergence structure. The example given here will be Banach limits.

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{C}$ be the collection of converging sequences in $\mathbb{K}$ and ℓ^∞ denote the collection of bounded sequence in $\mathbb{K}$. Using the Hahn-Banach theorem, one may construct a linear functional $L : \ell^\infty \to \mathbb{K}$ so that when $x \in \ell^\infty$

1. The operator norm of L is 1;
2. If $x \in \mathcal{C}$ then $L(x) = \lim x$;
3. If $x(n) \geq 0$ for all $n \in \mathbb{N}$ then $L(x) \geq 0$;
4. $L(T(x)) = L(x)$ where $T : \ell^\infty \to \ell^\infty$ is given by $T(x)(n) = x(n+1)$.

For details, see Theorem 7.1 of [Con90]. Such a functional is called a *Banach limit*. The usual limit operator is extended in such a way as to enable the following computation.

Example 2.1.13. Let $x \in \ell^\infty$ be given by $x = (1, 0, 1, 0, 1, ...)$ and $L : \ell^\infty \to \mathbb{K}$ be a Banach limit. The sequence x is certainly bounded but not convergent in the usual sense. Note that $T(x) = (0, 1, 0, 1, 0, ...)$. We thus have that

$$\begin{aligned} 1 &= \lim(1,1,1,1,...) \\ &= L(1,1,1,1,...) \\ &= L(x + T(x)) \\ &= L(x) + L(T(x)) \\ &= L(x) + L(x) \\ &= 2L(x) \end{aligned}$$

so that $L(x) = 1/2$.

Note that the sequence x above contains a constant subsequence at 0. If Banach limits could be given by a convergence structure, one would have to have $L(0) = 1/2$ which is clearly not possible.

Now that convergence spaces have been defined, we can formulate a definition of continuous maps between convergence spaces.

Definition 2.1.14. A function $f : X \to Y$ between convergence spaces X and Y is called *continuous at the point* $x \in X$ when for all filters $\mathcal{F} \to x$ in X one has $f(\mathcal{F}) \to f(x)$. If f is continuous at each point in its domain, it is said to be *continuous*. As in topology, a continuous bijection with continuous inverse is called a *homeomorphism*.

Unsurprisingly, continuity may be easily characterized by nets.

Proposition 2.1.15. *A function $f : X \to Y$ between convergence spaces is continuous at $x \in X$ if and only if for each net $\alpha \to x$ one has $f(\alpha) \to f(x)$.*

Proof. Suppose $f : X \to Y$ is continuous at some $x \in X$. Let α be a net in X with $\alpha \to x$. We then have that $\mathcal{E}(\alpha) \to x$ and by continuity that $f(\mathcal{E}(\alpha)) \to f(x)$. Since $f(\mathcal{E}(\alpha)) = \mathcal{E}(f(\alpha))$, we have that $f(\alpha) \to f(x)$.

Suppose $x \in X$ is such that for each net $\alpha \to x$ one has $f(\alpha) \to f(x)$. Fix a filter $\mathcal{F} \to x$. We then have that $\eta(\mathcal{F}) \to x$ and $f(\eta(\mathcal{F})) \to f(x)$. Since $f(\eta(\mathcal{F})) \sim \eta(f(\mathcal{F}))$, we have that $\eta(f(\mathcal{F})) \to f(x)$. Therefore, $f(\mathcal{F}) \to f(x)$ as desired for continuity of f at x. QED

The net characterization of continuity makes it plainly apparent that continuous maps between convergence spaces generalize sequentially continuous maps in metric and topological spaces.

We now have a notion of convergence spaces and maps between such spaces which interact with convergence structures in a reasonable way. We would like to say they form a category. For this, it only remains to prove the following result:

Proposition 2.1.16. *If X, Y and Z are convergence spaces and $f : X \to Y$ and $g : Y \to Z$ are continuous, then*

1. *The identity function $\mathrm{id}_X : X \to X$ which is given by $x \mapsto x$ is continuous;*

2. *The composition $g \circ f$ is continuous.*

Proof. Statement (1) is trivial. Certainly if a filter $\mathcal{F}$ converges in X, then so must $\mathrm{id}_X(\mathcal{F}) = \mathcal{F}$ and to the same point.

Consider now claim (2). Suppose $x \in X$ and we have some filter $\mathcal{F} \to x$. Recall from Corollary 1.5.6 that $g \circ f(\mathcal{F}) = g(f(\mathcal{F}))$. By continuity of f, we have that $f(\mathcal{F}) \to f(x)$. By continuity of g, we have $g(f(\mathcal{F})) \to g(f(x))$. Thus, $g \circ f(\mathcal{F}) \to g \circ f(x)$. Therefore, $g \circ f$ is continuous. QED

We now may safely define a category of convergence spaces.

Definition 2.1.17. The category **CONV** is that whose objects are convergence spaces and whose morphisms are continuous functions. If X and Y are convergence spaces, we write $C(X, Y)$ for the set of continuous functions from X to Y.

2.2 Relation to Topological Spaces

In the last section, we saw that every topological space gives rise to a convergence space via equipping the underlying set of the topological space with the filter convergence induced by the topology. If X is a topological space, we denoted the convergence space obtained in this manner by $\mathfrak{C}(X)$. A reasonable question then is whether it is possible, given convergence space X, to construct a topological space $\mathfrak{T}(X)$ which is in some way close to X. The answer here is affirmative.

In this section, we will show that $\mathfrak{C}$, though at this point purely symbolic, may be seen as a functor $\mathfrak{C}$: **TOP** $\to$ **CONV** witnessing that the category of topological spaces is a full subcategory of **CONV**. We will further construct a functor $\mathfrak{T}$: **CONV** $\to$ **TOP** which is left adjoint to $\mathfrak{C}$.

Definition 2.2.1. The functor $\mathfrak{C}$: **TOP** $\to$ **CONV** is defined by

1. If X is a topological space, $\mathfrak{C}(X)$ is X with the convergence structure induced by its topology;

2. If X and Y are topological spaces and $f : X \to Y$ is continuous, then $\mathfrak{C}(f) : \mathfrak{C}(X) \to \mathfrak{C}(Y)$ by $x \mapsto f(x)$.

The following two propositions will justify that $\mathfrak{C}$ is in fact a functor and **TOP** a full subcategory of **CONV**.

Proposition 2.2.2. *If X and Y are topological spaces and $f : X \to Y$ is continuous, then $\mathfrak{C}(f) : \mathfrak{C}(X) \to \mathfrak{C}(Y)$ is continuous.*

Proof. Let $x \in X$ and suppose there is a filter $\mathcal{F} \to x$ in $\mathfrak{C}(X)$. We then have that $\mathcal{F} \supseteq \mathcal{N}_x$ and thus $f(\mathcal{F}) \supseteq f(\mathcal{N}_x)$. Suppose N is a neighborhood of $f(x)$. We then have by continuity of f that $f^{-1}(N)$ is a neighborhood of x. This of course means $f^{-1}(N) \in \mathcal{N}_x$. We have now

$$N \supseteq f(f^{-1}(N)) \in f(\mathcal{N}_x)$$

which tells us $N \in f(\mathcal{N}_x)$. Therefore, $f(\mathcal{N}_x) \supseteq \mathcal{N}_{f(x)}$. By definition, we then have that $f(\mathcal{N}_x) \to f(x)$ and thus that $\mathfrak{C}(f)$ is continuous. QED

Proposition 2.2.3. *Suppose X and Y are topological spaces. If $f : \mathfrak{C}(X) \to \mathfrak{C}(Y)$ is a continuous mapping of convergence spaces, then $f : X \to Y$ is also a continuous mapping of topological spaces.*

Proof. Fix $x \in X$. We have that $\mathcal{N}_x \to x$, and thus by continuity of f we know $f(\mathcal{N}_x) \to f(x)$. This implies $f(\mathcal{N}_x) \supseteq \mathcal{N}_{f(x)}$. Thus, for every neighborhood U of $f(x)$, there is some neighborhood N of x so that $f(N) \subseteq U$. Therefore, f is continuous at x for all $x \in X$. QED

So, $\mathfrak{C}$ is a functor embedding **TOP** into **CONV** as a full subcategory. From here on, if X is a topological space, we will often not distinguish symbolically between X the topological space and $\mathfrak{C}(X)$ the convergence space. Likewise, if f is a continuous mapping of topological spaces, we will often write simply f for $\mathfrak{C}(f)$. Explicit use of $\mathfrak{C}$ will occur when the context demands extra care.

We will now introduce the set-up required to define the functor $\mathfrak{T}$: **CONV** $\to$ **TOP**.

Definition 2.2.4. If X is a convergence space and $x \in X$, we define the *vicinity filter* at x as

$$\mathcal{V}_x = \bigcap \{\mathcal{F} \in \Phi(X) : \mathcal{F} \to x\}.$$

Elements of $\mathcal{V}_x$ are called *vicinities* of x. A subset of a convergence space is called *open* when it is a vicinity of each of its points and *closed* when its complement is open.

Remark 2.2.5. Recall that a filter $\mathcal{F}$ converges to a point in a topological space if and only if $\mathcal{F}$ contains all neighborhoods of this point. Thus, for a topological space X, the vicinity filter at a point in $\mathfrak{C}(X)$ is just the neighborhood filter at that point in X. Thus, the open sets of $\mathfrak{C}(X)$ are exactly the open sets of X.

Remark 2.2.6. The term vicinity filter is non-standard. In [BB02] and [Pat14], these filters are called neighborhood filters. This name serves as a reminder that vicinities and neighborhoods coincide for topological spaces and extends the topological fact that a set is open if and only if it is a neighborhood of each of its points. However, this term can be misleading. Consider Example 2.43 of [Pat14]: Give $X = \{0,1,2\}$ a convergence structure by

$$\mathcal{F} \to 0 \text{ iff } \{0,1\} \in \mathcal{F}$$
$$\mathcal{F} \to 1 \text{ iff } \{1,2\} \in \mathcal{F}$$
$$\mathcal{F} \to 2 \text{ iff } \{0,2\} \in \mathcal{F}$$

With this structure, one has that

$$\mathcal{V}_0 = [\{0,1\}]$$
$$\mathcal{V}_1 = [\{1,2\}]$$
$$\mathcal{V}_2 = [\{0,2\}]$$

and that the only open sets are $\emptyset$ and X. As we will soon prove is always the case, the open subsets of X form a topology, but the neighborhood filters of points in this topology are not the vicinity filters giving rise to the topology.

In [Nel16], vicinity filters are called assembled filters. This terminology offers no confusion with usual topological notions. However, it seems too divorced from the topological motivation for vicinity filters. Thus, as in [DM16], the term vicinity is chosen to retain the flavor of neighborhoods but not invite confusion.

We can reformulate the above definition of open sets as a property which is easier to verify in practice.

Proposition 2.2.7. *A subset V of a convergence space X is a vicinity of $x \in X$ if and only if either of the equivalent conditions are met:*

1. *for every filter $\mathcal{F} \to x$ we have $V \in \mathcal{F}$;*
2. *for every net $\alpha \to x$ we have $\alpha \in_{ev} V$.*

Proof. Suppose $x \in X$ and V is a vicinity of x. Consider a filter $\mathcal{F} \to x$. We then have by the definition of vicinity filter that $\mathcal{F} \supseteq \mathcal{V}_x$. But then certainly, we have that $V \in \mathcal{F}$. Likewise, if α is a net converging to x, then $\mathcal{E}(\alpha) \to x$. By the above reasoning, we have $V \in \mathcal{E}(\alpha)$. This is exactly the statement that $\alpha \in_{\text{ev}} V$.

Now, suppose V satisfies (1). This says

$$V \in \bigcap \{\mathcal{F} \in \Phi(X) : \mathcal{F} \to x\} = \mathcal{V}_x.$$

Thus, V is a vicinity of x. Now, suppose V satisfies (2). Consider a filter $\mathcal{F} \to x$. We have that $\eta(\mathcal{F}) \to x$. Thus, $\eta(\mathcal{F}) \in_{\text{ev}} V$. Therefore,

$$V \in \mathcal{E}(\eta(\mathcal{F})) = \mathcal{F}.$$

This holds for any filter converging to x, so V is a vicinity of x. QED

The terminology here is suggestive; it would be ideal if the open subsets of a convergence space formed a topology.

Theorem 2.2.8. *If X is a convergence space, the collection of open subsets of X form a topology.*

Proof. Observe that every converging filter in X contains X. So X is open. Further, we have that $\emptyset$ is open by vacuity.

Suppose $\mathcal{U}$ is a family open open subsets of X. Suppose $x \in \bigcup \mathcal{U}$ and $\mathcal{F}$ is a filter converging to x. Then $x \in U$ for some $U \in \mathcal{U}$. Since U is open we have that $U \in \mathcal{F}$. Since $\bigcup \mathcal{U} \supseteq U$, we have that $\bigcup \mathcal{U} \in \mathcal{F}$. This holds for all $x \in \bigcup \mathcal{U}$ and filters $\mathcal{F} \to x$, we have that $\bigcup \mathcal{U}$ is open.

Now, suppose $U_1, ..., U_n$ is a family of open subsets of x. Define $U = U_1 \cap \cdots \cap U_n$. Suppose $x \in U$ and $\mathcal{F}$ is a filter converging to x. We have that $x \in U_1, ..., U_n$. Each is open, so $U_1, ..., U_n \in \mathcal{F}$. Since $\mathcal{F}$ is a filter, it is closed under finite intersections. Therefore, $U \in \mathcal{F}$. Therefore, U is a vicinity of each of its points and is open.

The collection of open subsets of X include X and $\emptyset$ and are closed under arbitrary unions and finite intersections. This is precisely that they form a topology. QED

If X is a convergence space, we denote the topological space with underlying set X and open sets given by the convergence structure by $\mathfrak{T}(X)$. We next show that the assignment $X \mapsto \mathfrak{T}(X)$ is functorial.

Proposition 2.2.9. *If X and Y are convergence spaces and $f : X \to Y$ is a continuous function, the map $\mathfrak{T}(f) : \mathfrak{T}(X) \to \mathfrak{T}(Y)$ given by $x \mapsto f(x)$ is continuous as a mapping of topological spaces.*

Proof. Suppose $U \subseteq \mathfrak{T}(Y)$ is open. Fix $x \in f^{-1}(U)$. Suppose α is a net converging to x in X. By continuity of f as a mapping of convergence spaces, we have that $f(\alpha) \to f(x)$. Therefore, $f(\alpha) \in_{\text{ev}} U$. We thus have that $\alpha \in_{\text{ev}} f^{-1}(U)$. So $f^{-1}(U)$ is a vicinity of each of its points. It is therefore open as a subset of X. By definition, it is open in $\mathfrak{T}(X)$. QED

Corollary 2.2.10. *If X and Y are convergence spaces and $f : X \to Y$ is continuous, then for any open $U \subseteq Y$ we have $f^{-1}(U)$ open in X.*

We interpret $\mathfrak{T}$ as a functor $\mathfrak{T} : \textbf{CONV} \to \textbf{TOP}$. For any convergence space X, by definition of $\mathfrak{T}$ and Remark 2.2.5, we have that $\mathfrak{T}\mathfrak{C}(X) = X$. Certainly for any continuous mapping f of topological spaces, we then have $\mathfrak{T}\mathfrak{C}(f) = f$. Thus, as functors, $\mathfrak{T}\mathfrak{C} = \text{id}_{\textbf{TOP}}$.

In fact, the functors $\mathfrak{T}$ and $\mathfrak{C}$ form an adjunction.

Notation 2.2.11. If X is a convergence space with convergence structure $\to$, denote the convergence structure in $\mathfrak{C}\mathfrak{T}(X)$ by $\to_\tau$.

Theorem 2.2.12. *If X is a convergence space, the map $\tau_X : X \to \mathfrak{C}\mathfrak{T}$ given by $x \mapsto x$ is continuous.*

Proof. Suppose α is a net in X converging to $x \in X$. Then for any open set $U \ni x$ in X we have that $\alpha \in_{\text{ev}} U$. But this is exactly that α is eventually in each open set containing x in $\mathfrak{C}\mathfrak{T}(X)$. This is exactly that $\alpha \to_\tau x$. Therefore, τ_X is continuous. QED

Lemma 2.2.13. *The functor $\mathfrak{T}$ is left adjoint to $\mathfrak{C}$.*

Proof. We note that for any convergence spaces X and Y and any continuous $f : X \to Y$, we have that the diagram

$$\begin{CD} X @>{f}>> Y \\ @V{\tau_X}VV @VV{\tau_Y}V \\ \mathfrak{C}\mathfrak{T}(X) @>>{\mathfrak{C}\mathfrak{T}(f)}> \mathfrak{C}\mathfrak{T}(Y) \end{CD}$$

commutes. Thus, the maps τ_X are the components of a natural transformation $\tau : \text{id}_{\textbf{CONV}} \to \mathfrak{C}\mathfrak{T}$. Further, since $\mathfrak{T}\mathfrak{C} = \text{id}_{\textbf{TOP}}$, we have a trivial natural transformation $\text{id} : \mathfrak{T}\mathfrak{C} \to \text{id}_{\textbf{TOP}}$. These natural transformations are such that for any convergence spaces X and Y, the diagrams

$$\begin{array}{ccc} \mathfrak{T}(X) & \xrightarrow{\mathfrak{T}(\tau_X)} & \mathfrak{T}\mathfrak{C}\mathfrak{T}(X) \\ & \searrow^{\text{id}_{\mathfrak{T}(X)}} & \downarrow{\scriptstyle \text{id}_{\mathfrak{T}(X)}} \\ & & \mathfrak{T}(X) \end{array} \qquad \begin{array}{ccc} \mathfrak{C}(Y) & \xrightarrow{\tau_{\mathfrak{C}(Y)}} & \mathfrak{C}\mathfrak{T}\mathfrak{C}(X) \\ & \searrow^{\text{id}_{\mathfrak{C}(Y)}} & \downarrow{\scriptstyle \mathfrak{C}(\text{id}_Y)} \\ & & \mathfrak{C}(Y) \end{array}$$

commute. Thus, $\mathfrak{T}$ and $\mathfrak{C}$ form an adjoint pair with unit and counit of adjunction τ and id respectively. QED

Corollary 2.2.14. *The functor $\mathfrak{T}$ is cocontinuous and $\mathfrak{C}$ is continuous.*

Corollary 2.2.15. *For any convergence space X and any topological space Y and any continuous function $f : X \to \mathfrak{C}(Y)$ there is a unique continuous map $\widehat{f} : \mathfrak{C}\mathfrak{T}(X) \to \mathfrak{C}(Y)$ so that*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & \mathfrak{C}(Y) \\ & {\scriptstyle \tau_X}\searrow \quad \nearrow{\scriptstyle \widehat{f}} & \\ & \mathfrak{C}\mathfrak{T}(X) & \end{array}$$

commutes.

Proof. Since X and $\mathfrak{T}\mathfrak{C}(X)$ have the same underlying set, and τ_X is continuous, it suffices to show that f is continuous as a map out of $\mathfrak{C}\mathfrak{T}(X)$.

We have that $\mathfrak{T}(f) : \mathfrak{T}(X) \to \mathfrak{T}\mathfrak{C}(Y) = Y$ is continuous. Then $\mathfrak{C}\mathfrak{T}(f) : \mathfrak{C}\mathfrak{T}(X) \to \mathfrak{C}(Y)$. This is the desired result. QED

Corollary 2.2.16. *A convergence space X is topological if and only if there is a topological space Y so that $X \cong \mathfrak{C}(Y)$.*

Proof. Suppose X is a topological convergence space. Recall that by definition, X is topological when there is a topological space Y with $X = \mathfrak{C}(Y)$. Then certainly $X \cong \mathfrak{C}(Y)$.

For the other direction, suppose there is a topological space Y and homeomorphism $f : X \to \mathfrak{C}(Y)$. We then have the homeomorphism $\mathfrak{C}\mathfrak{T}(f) : \mathfrak{C}\mathfrak{T}(X) \to \mathfrak{C}(Y)$ with $\mathfrak{C}\mathfrak{T}(f) \circ \tau_X = f$. Since $\mathfrak{C}\mathfrak{T}(f)$ and f are homeomorphisms, so is τ_X. Thus, $X = \mathfrak{C}\mathfrak{T}(X)$ and X is a topological convergence space. QED

Remark 2.2.17. In Corollary 2.2.10, we show that the preimage of open sets under continuous mappings of convergence spaces are open. This is what one expects from topology. However, the converse fails. Note X and $\mathfrak{C}\mathfrak{T}(X)$ share the same open sets for any convergence space X. Thus, the preimage of any opens set under τ_X^{-1} is open. However, τ_X^{-1} is not continuous unless X is a topological convergence space.

Above, we saw how the convergence of filters and nets describe the open subsets of a convergence space. Convergence can also be used to detect closed sets. We first need a definition.

Definition 2.2.18. If S is a subset of a convergence space X, we define the *adherence* of S as

$$a(S) = \{x \in X : \exists \mathcal{F} \in \Phi(X)(\mathcal{F} \to x \text{ and } S \in \mathcal{F})\}.$$

The adherence of course has an equivalent characterization via nets.

Proposition 2.2.19. *If S is a subset of a convergence space X, then*

$$a(S) = \{x \in X : \textit{ there is a net } \alpha \textit{ in } S \textit{ with } \alpha \to x\}.$$

Proof. Suppose $x \in a(S)$. There is then a filter $\mathcal{F} \to x$ with $S \in \mathcal{F}$. We thus have that $\eta(\mathcal{F}) \to x$ and $\eta(\mathcal{F}) \in_{\text{ev}} S$. We may thus choose a subnet α of $\eta(\mathcal{F})$ so that α is entirely in S.

Now, suppose that there is a net α in S with $\alpha \to x$ for some $x \in X$. Then $\mathcal{E}(\alpha) \to x$ and $S \in \mathcal{E}(\alpha)$. So $x \in a(S)$. We conclude

$$a(S) = \{x \in X : \text{ there is a net } \alpha \text{ in } S \text{ with } \alpha \to x\}.$$

as desired. QED

Proposition 2.2.20. *If $f : X \to Y$ is a continuous mapping of convergence spaces, and $S \subseteq X$, then $f(a(S)) \subseteq a(f(S))$.*

Proof. Suppose $y \in f(a(S))$. There is then a net α in S and $x \in X$ so that $\alpha \to x$ and $f(x) = y$. Then, $f(\alpha)$ is a net in $f(S)$ and $f(\alpha) \to y$ by continuity of f. Thus, $y \in a(f(S))$. We conclude $f(a(S)) \subseteq a(f(S))$ as desired. QED

We may now classify closed sets.

Lemma 2.2.21. *For any convergence space X and subset $S \subseteq X$ we have $S \subseteq a(S)$.*

Proof. For each $x \in S$ we have $[x] \to x$ and $S \in [x]$. Therefore, $x \in a(S)$ and $S \subseteq a(S)$. QED

Proposition 2.2.22. *If S is a subset of a convergence space X, then the following are equivalent:*

1. *S is closed,* i.e. *$X \setminus S$ is open;*

2. *$a(S) = S$.*

Proof. Suppose $X \setminus S$ is open. Let $x \in a(S)$. There is then a filter $\mathcal{F} \to x$ with $S \in \mathcal{F}$. But then $X \setminus S \notin \mathcal{F}$. So $X \setminus S$ is not a vicinity of x. Since $X \setminus S$ is open, we have $x \notin X \setminus S$. Thus, $x \in S$ and $a(S) \subseteq S$ and $a(S) = S$.

Now, suppose $a(S) = S$ and $x \in X \setminus S$ and $\alpha \to x$ for some filter α. Suppose there is a cofinite subset I of $\mathrm{dom}(\alpha)$ so that $\alpha(I) \subseteq S$. Then we may define a subnet of α entirely contained in S. But then $x \in a(S) = S$ which is impossible. Thus, $\alpha \in_{\text{ev}} X \setminus S$. Thus, $X \setminus S$ is a vicinity of x for each $x \in X \setminus S$. So, $X \setminus S$ is open and S closed. QED

Proposition 2.2.23. *If X is a convergence space and $S \subseteq T \subseteq X$, then $a(S) \subseteq a(T)$.*

Proof. This follows immediately since any filter on X containing S contains T. QED

Corollary 2.2.24. *If X is a convergence space and $S \subseteq X$, then $a(S) \subseteq \overline{S}$, the topological closure of S.*

Proposition 2.2.25. *If $f : X \to Y$ is a continuous mapping of convergence spaces, then $f(\mathcal{V}_x) \supseteq \mathcal{V}_{f(x)}$ for all $x \in X$.*

Proof. Fix $x \in X$ and $V \in \mathcal{V}_{f(x)}$. For any filter $\mathcal{F} \to x$ in X, we have that $f(\mathcal{F}) \to f(x)$ and thus $f^{-1}(V) \in \mathcal{F}$. Thus, $f^{-1}(V) \in \mathcal{V}_x$ and $V \in f(\mathcal{V}_x)$. Thus, $f(\mathcal{V}_x) \supseteq \mathcal{V}_{f(x)}$ as desired. QED

Corollary 2.2.26. *The preimage of a vicinity of $f(x)$ under a continuous mapping f is a vicinity of x.*

In summary, we have seen that the category of topological spaces is a full subcategory of the category of convergence spaces. We further have a left adjoint to this inclusion which produces from any convergence space a topological space which is "close" to the convergence space in some sense.

2.3 New Convergence Structures from Old

In this section, we introduce the initial and final convergence structures. These allow us to produce new convergence spaces from old ones. The initial structure allows us to define such objects as subspaces, products, and weak topologies. The final structure gives quotients, coproducts, and other such objects.

2.3.1 Initial Convergence Structure

Definition 2.3.1. Suppose X is a set and $\{f_i : X \to X_i : i \in I\}$ is a collection of functions from X to convergence spaces X_i. The *initial convergence structure* on X with respect to $\{f_i : X \to X_i : i \in I\}$ is given by

$$\mathcal{F} \to x \iff f_i(\mathcal{F}) \to f_i(x) \text{ for all } i \in I$$

for filters $\mathcal{F} \in \Phi(X)$ and points $x \in X$.

Remark 2.3.2. It is clear from this definition that if X is a set and $\{f_i : X \to X_i : i \in I\}$ is a collection of functions from X to convergence spaces X_i, then a net α in X converges to $x \in X$ precisely when $f_i(\alpha) \to f_i(x)$ for all $i \in I$.

Proposition 2.3.3. *If X is a set and $\{f_i : X \to X_i : i \in I\}$ is a collection of functions from X to convergence spaces X_i, the initial convergence structure on X with respect to the f_i is actually a convergence structure.*

Proof. We need merely check the defining characteristics of a convergence space. Fix $x \in X$.

For each $i \in I$, we have $f_i([x]) = [f_i(x)] \to f_i(x)$ so that $[x] \to x$.

Suppose $\mathcal{F}, \mathcal{G} \in \Phi(X)$ both converge to x. Then we have $f_i(\mathcal{F}) \to f_i(x)$ and $f_i(\mathcal{G}) \to f_i(x)$ for each $i \in I$. Thus,

$$f_i(\mathcal{F} \cap \mathcal{G}) = f_i(\mathcal{F}) \cap f_i(\mathcal{G}) \to f_i(x)$$

for each $i \in I$ so that $\mathcal{F} \cap \mathcal{G} \to x$.

Suppose $\mathcal{F}, \mathcal{G} \in \Phi(X)$ with $\mathcal{G} \supseteq \mathcal{F}$ and $\mathcal{F} \to x$. Then

$$f_i(\mathcal{G}) \supseteq f_i(\mathcal{F}) \to f_i(x)$$

so that $f_i(\mathcal{G}) \to f_i(x)$ for each $i \in I$. Thus, $\mathcal{G} \to x$.

We thus have that the initial convergence structure is a convergence structure. QED

Remark 2.3.4. The maps f_i giving the initial convergence structure are continuous when X carries the initial convergence structure.

The initial convergence structure comes with an important universal property.

Proposition 2.3.5. *If X carries the initial convergence structure over a family $\{f_i : X \to X_i : i \in I\}$ and Y is any convergence space, a function $f : Y \to X$ is continuous if and only if $f_i \circ f$ is continuous for each $i \in I$.*

Proof. Certainly, if f is continuous, then $f_i \circ f$ is continuous for each $i \in I$ as a composition of continuous functions.

Suppose $f_i \circ f$ is continuous for each $i \in I$. Suppose $y \in Y$ and α is a net in Y with $\alpha \to y$. Then by continuity, $f_i \circ f(\alpha) \to f_i(f(y))$ for each $i \in I$. But this is exactly that $f(\alpha) \to f(y)$ in the initial convergence structure. QED

Remark 2.3.6. This universal property completely characterizes the initial convergence structure; that is, the initial convergence structure on X is the only one satisfying this universal property.

Proposition 2.3.7. *Every convergence space X carries the initial convergence structure over $C(X, X)$.*

Proof. If a filter $\mathcal{F}$ converges in X, then so does $f(\mathcal{F})$ for every $f \in C(X, X)$. Further, if a point $x \in X$ and filter $\mathcal{F}$ on X is such that $f(\mathcal{F}) \to f(x)$ for every $f \in C(X, X)$, then in particular $\mathcal{F} = \mathrm{id}_X(\mathcal{F}) \to x$. Thus, convergence in X is exactly convergence in the initial structure over $C(X, X)$. QED

We now define three important instances of the initial convergence structure.

Definition 2.3.8. Suppose X is a convergence space and $S \subseteq X$. The *subspace convergence structure* on S is the initial convergence structure with respect to the inclusion $\iota : S \hookrightarrow X$.

Definition 2.3.9. If X, Y are convergence spaces and $f : Y \to X$ is a homeomorphism onto its image (with the subspace convergence structure), then f is called an *embedding*.

Definition 2.3.10. If $\{X_i : i \in I\}$ is a collection of convergence spaces, the initial convergence structure on $X = \prod_{i \in I} X_i$ with respect to the projections $\pi_i : X \to X_i$ is called the *product convergence structure*. Unless otherwise stated, all products of convergence spaces will be assumed to carry this convergence structure.

Definition 2.3.11. If X is a convergence space, the initial convergence structure on X with respect to $C(X, \mathbb{K})$ is called the *weak convergence structure* or *weak topology* on X. A space X with this convergence structure is denoted X_σ and convergence therein is denoted $\to_\sigma$. If $S \subseteq X$, we denote the closure of S in the weak topology by $\overline{S}^\sigma$

Remark 2.3.12. We call the weak convergence structure the weak topology precisely because it is a topological convergence structure whose underlying topology is the weak topology. This will be established by Proposition 2.6.5 and Definition 2.6.9.

With these definitions in place, we will prove some basic properties.

Given the name, one hopes that the subspace convergence structure is related to the usual subspace topology. This is the case.

Proposition 2.3.13. *If X is a topological space and $S \subseteq X$ carries the subspace topology, then $\mathfrak{C}(S)$ carries the subspace convergence structure with respect to $\mathfrak{C}(X)$.*

Proof. We certainly have that the inclusion : $\mathfrak{C}(S) \hookrightarrow \mathfrak{C}(X)$ is continuous by functoriality of $\mathfrak{C}$.

Now, suppose Y is some convergence space and $f : Y \to \mathfrak{C}(S)$ is some function so that $i \circ f$ is continuous. Let $y \in Y$ and $U \subseteq S$ an open set containing $f(y)$. Let α be a net in Y with $\alpha \to y$. There is then some open set $V \subseteq X$ with $U = S \cap V$. Since $i \circ f$ is continuous, we have $i \circ f(\alpha) \to i \circ f(y) = f(y)$. So, $f(\alpha) \in_{\text{ev}} V$. Since $f(\alpha)$ is a net in S, we have $f(\alpha) \in_{\text{ev}} U$. Thus, $f(\alpha) \to f(y)$.

Thus, the convergence structure on $\mathfrak{C}(S)$ satisfies the universal property of the subspace convergence structure and this is the subspace convergence structure. QED

The following is a useful technical detail.

Proposition 2.3.14. *Fix a convergence space X and filter $\mathcal{F}$ on X converging to some $x \in X$. If $x \in S \subseteq X$ is such that $S \cap F \neq \emptyset$ for all $F \in \mathcal{F}$, then $\mathcal{F}|_S \to x$ in S with the subspace convergence structure. Additionally, if $S \in \mathcal{F}$, then $\mathcal{F}|_S \to y$ in S implies $\mathcal{F} \to y$ in X.*

Proof. Recalling the discussion in Notation 1.3.9, we have that

$$[\mathcal{F}|_S] = \mathcal{F} \cap S \supseteq \mathcal{F}.$$

Since $\mathcal{F} \to x$, we have $[\mathcal{F}|_S] \to x$ in X. Thus, $\mathcal{F}|_S \to x$ is S.

For the rest of the proof, it suffices to remark that if $S \in \mathcal{F}$, then $\mathcal{F} \cap S = \mathcal{F}$. QED

In topology, it is often possible to define a continuous map on a space X by first specifying it on subspaces and then gluing this data together to form a function on the whole space. This can also be done for convergence spaces

Proposition 2.3.15 (General Gluing Lemma)**.** *Let Y be a convergence space. If X is a convergence space satisfying*

1. *there is a finite collection $\mathcal{A}$ of subsets of X with $\bigcup \mathcal{A} = X$;*
2. *for each $A \in \mathcal{A}$ there is a continuous function $f_A : A \to Y$;*
3. *for each $A, B \in \mathcal{A}$ and $x \in A \cap B$ we have $f_A(x) = f_B(x)$;*
4. *for each $A, B \in \mathcal{A}$ and every net α in A so that $\alpha \to x$ for some $x \in B$ we have $f_A(\alpha) \to f_B(x)$;*

then there is continuous $h : X \to Y$ so that for each $A \in \mathcal{A}$ we have $h|_A = f_A$.

Proof. Letting $\mathcal{A} = \{A_1, ..., A_n\}$ we define $f : X \to Y$ by

$$h(x) = \begin{cases} f_{A_1}(x) & x \in A_1 \\ f_{A_2}(x) & x \in A_2 \\ \vdots & \vdots \\ f_{A_n}(x) & x \in A_n \end{cases}$$

which is well defined by (1) and (3).

Suppose $x \in X$ and there is a net $\alpha \to x$. Let D be the domain of α and for each $k = 1, ..., n$ define $D_k = \alpha^{-1}(A_k)$. Let

$$\mathcal{B} = \{A_k : D_k \text{ cofinal in } D\}.$$

We have that for each k so that $A_k \in \mathcal{B}$ that $\alpha|_{D_k} \to x$. Without loss of generality, let $x \in A_1$. For each k so that $A_k \in \mathcal{B}$

$$h(\alpha|_{D_k}) = f_{A_k}(\alpha|_{D_k}) \to f_{A_1}(x) = h(x)$$

by continuity of f_{A_1} and (4). Produce $d_0 \in D$ so that for all $d \geq d_0$ we have $\alpha_d \in \bigcup \mathcal{B}$. It follows that $D_{\geq d_0} \subseteq \bigcup \{D_k : A_k \in \mathcal{B}\}$. By Proposition 2.1.8 we have that $h(\alpha) \to h(x)$ so that h is continuous as desired. QED

Corollary 2.3.16 (Open Set Gluing Lemma)**.** *Suppose X, Y are convergence spaces, and $U, V \subseteq X$ are open with $U \cup V = X$. If $f : U \to Y$ and $g : V \to Y$ are continuous functions agreeing on $U \cap V$, then $h : X \to Y$ by*

$$h(x) = \begin{cases} f(x) & x \in U \\ g(x) & x \in V \end{cases}$$

is continuous.

Proof. We need only check the hypotheses of the general gluing lemma, of which only (4) requires any work. Without loss of generality, suppose there is a net α in U converging to $x \in V$. Since V is open, $\alpha \in_{\text{ev}} V$. Therefore, $\alpha \in_{\text{ev}} U \cap V$. Since f and g agree on $U \cap V$ and g is continuous, we have $f(\alpha) \to g(x)$ as desired. QED

Corollary 2.3.17. *Suppose $\mathcal{U}$ is a finite collection of open subsets of a convergence space X and $X = \bigcup \mathcal{U}$. Suppose Y is any convergence space. If for each $U \in \mathcal{U}$ there is continuous $f_U : U \to Y$ so that for any $U, V \in \mathcal{U}$ we have $f_U(x) = f_V(x)$ for all $x \in U \cap V$, then there is a continuous function $h : X \to Y$ so that for all $x \in X$ and $U \in \mathcal{U}$ we have $h(x) = f_U(x)$ whenever $x \in U$.*

We now will turn our attention to the product of convergence spaces. The main result here is that the product of convergence spaces is the categorical product in **CONV**.

Proposition 2.3.18. *If $\{X_i : i \in I\}$ is a family of convergence spaces, then $\prod_{i \in I} X_i$ is the categorical product of spaces X_i.*

Proof. This is an immediate consequence of Proposition 2.3.5 and the fact that the Cartesian product is the categorical product in SET. QED

Corollary 2.3.19. *If $\{X_i\}$ is a family of topological spaces,*

$$\prod_{i \in I} \mathfrak{C}(X_i) = \mathfrak{C}\Big(\prod_{i \in I} X_i\Big).$$

Proof. The follows from the continuity of $\mathfrak{C}$ as established in Corollary 2.2.14 QED

Corollary 2.3.20. *The product of topological convergence spaces is again topological.*

We end with a useful result saying roughly that to study convergence in a product space, it is enough to understand the products of filters on the factor spaces.

Proposition 2.3.21. *If X and Y are convergence spaces, a filter $\mathcal{H}$ on $X \times Y$ converges to (x, y) if and only if there are filters $\mathcal{F} \to x$ and $\mathcal{G} \to y$ in X and Y respectively so that $\mathcal{H} \supseteq \mathcal{F} \times \mathcal{G}$.*

Proof. This is an immediate consequence of Proposition 1.6.14. QED

Lastly, we address an apparent ambiguity in the definition of weak convergence structure, the value of the ground field $\mathbb{K}$.

Proposition 2.3.22. *If X is any convergence space, then the initial convergence structures on X with respect to $C(X, \mathbb{R})$ and $C(X, \mathbb{C})$ are identical.*

Proof. Let X_σ denote X with the initial convergence structure over $C(X, \mathbb{C})$. We will prove the desired claim by showing that if Y is a convergence space then a function $f_0 : Y \to X_\sigma$ is continuous if and only if $f \circ f_0$ is continuous for each $f \in C(X, \mathbb{R})$.

Suppose f_0 continuous and $f \in C(X, \mathbb{R})$ is continuous. Let $i : \mathbb{R} \hookrightarrow \mathbb{C}$ be the usual embedding. We have that $i \circ f \in C(X, \mathbb{C})$. Thus, $i \circ f \circ f_0$ is continuous. Let $\pi_1 : \mathbb{C} \to \mathbb{R}$ take $z \in \mathbb{C}$ to its real part. We have $\pi_1 \circ i \circ f \circ f_0 = f \circ f_0$ is continuous.

Now, suppose $f \circ f_0$ is continuous for each $f \in C(X, \mathbb{R})$. Let $h \in C(X, \mathbb{C})$. We may write $h = h_1 + ih_2$ for continuous real valued h_1, h_2. Thus, $h \circ f_0 = h_1 \circ f_0 + i(h_2 \circ f_0)$ is continuous and by the universal property of initial convergence structures, f_0 is continuous.

Thus, X_σ satisfies the universal property of X with the initial convergence structure over $C(X, \mathbb{R})$. We conclude the two spaces are the same. QED

2.3.2 Final Convergence Structure

Definition 2.3.23. Suppose X is a set and $\{f_i : X_i \to X \mid i \in I\}$ is some family of functions out of convergence spaces X_i. The *final convergence structure* on X relative to the f_i is given by

> $\mathcal{F} \to x$ iff $\mathcal{F} = [x]$ or there is a finite finite collection of indices $J \subseteq I$. So that for each $j \in J$ there is a finite subcollection $Z_j \subseteq X_j$ so that for each $z \in Z_j$ we have both $f_j(z) = x$ and a filter $\mathcal{F}_z \to z$ so that
>
> $$\mathcal{F} \supseteq \bigcap_{j \in J} \bigcap_{z \in Z_j} f_j(\mathcal{F}_z)$$

for all $x \in X$ and filters $\mathcal{F}$ on X.

One may check that this is in fact a convergence structure. Further, this convergence structure makes each f_i continuous.

Proposition 2.3.24. *Suppose X is given the final convergence structure relative to the family of maps $\{f_i : X_i \to X\}_{i \in I}$ out of convergence spaces X_i. If Ω is some convergence space, then a function $f : X \to \Omega$ is continuous if and only if $f \circ f_i$ is continuous for all $i \in I$.*

Proof. The case in which f is assumed continuous is trivial.

Assume that $f \circ f_i$ is continuous for all $i \in I$. Let $x \in X$ and $\mathcal{F} \to x$. If $\mathcal{F}$ is the point filter at x, then $f(\mathcal{F}) = [f(x)]$ which must converge. Else, assume that $\mathcal{F}$ is not the point filter. Then there is a finite collection of indices $J \subseteq I$ so that for each $j \in J$ there is a finite subcollection $Z_j \subseteq X_j$ such that for each $z \in Z_j$ we have both $f_j(z) = x$ and a filter $\mathcal{F}_z \to z$ so that

$$\mathcal{F} \supseteq \bigcap_{j \in J} \bigcap_{z \in Z_j} f_j(\mathcal{F}_z).$$

We apply f and obtain

$$f(\mathcal{F}) \supseteq \bigcap_{j \in J} \bigcap_{z \in Z_j} f \circ f_j(\mathcal{F}_z).$$

By continuity, each filter on the right hand side converges to $f(x)$. Therefore $f(\mathcal{F}) \to f(x)$ and we conclude that f is continuous. QED

There are two final convergence structures which are of particular interest. The first is the coproduct in **CONV**

Definition 2.3.25. If $\{X_i\}_{i \in I}$ is a family of convergence spaces and $X = \coprod_{i \in I} X_i$ is the disjoint unions of the underlying sets of the X_i, then X along with the final convergence structure relative to the standard embeddings $e_i : X_i \to X$ is called the *convergence sum*.

Using the fact that disjoint union is the coproduct in **SET** and Proposition 2.3.24 one has that the convergence sum is the coproduct in **CONV**.

Proposition 2.3.26. *If $\{X_i : i \in I\}$ is a family of topological spaces, then*

$$\coprod_{i\in I} \mathfrak{C}(X_i) = \mathfrak{C}\Big(\coprod_{i\in I} X_i\Big).$$

Proof. Since the embeddings $e_i : \mathfrak{C}(X_i) \to \mathfrak{C}\big(\coprod_{i\in I} X_i\big)$ are continuous, we have that the "identity" $\coprod_{i\in I} \mathfrak{C}(X_i) \to \mathfrak{C}\big(\coprod_{i\in I} X_i\big)$ is continuous. Thus, it only remains to show that if a net $\alpha \to x$ in $\mathfrak{C}\big(\coprod_{i\in I} X_i\big)$, then $\alpha \to x$ in $\coprod_{i\in I} \mathfrak{C}(X_i)$.

We observe that if a net $\alpha \to x$ in $\mathfrak{C}\big(\coprod_{i\in I} X_i\big)$, $x \in X_i$ for some $i \in I$ and without loss of generality, α is a net in X_i eventually in each neighborhood of x. So $\alpha \to x$ in $\mathfrak{C}(X_i)$. By continuity of the embeddings, $\alpha \to x$ in $\coprod_{i\in I} \mathfrak{C}(X_i)$ and we are done. QED

The second notable final structure is the quotient.

Definition 2.3.27. If X is a convergence space, Y is a set, and $q : X \to Y$ is a surjection, then the final convergence structure on Y relative to q is called the *quotient convergence structure*. The map q is then called the quotient map.

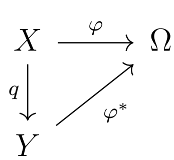
The condition for filter convergence given in Definition 2.3.23 becomes somewhat simpler when working with a quotient convergence structure.

Proposition 2.3.28. *Suppose Y is a set, X is a convergence space, and $f : X \to Y$ is a surjection. Place the quotient convergence structure relative to f on X. A filter $\mathcal{F}$ converges to some $y \in Y$ if and only if $(*)$ where $(*)$ is*

> *there exist $x_1, ..., x_n \in X$ and $\mathcal{F}_1, ..., \mathcal{F}_n \in \Phi(X)$ so that for each $i \in \{1, ..., n\}$ one has $\mathcal{F}_i \to x_i$ and $f(x_i) = y$ and*
>
> $$\mathcal{F} \supseteq \bigcap_{i=1}^{n} f(\mathcal{F}_i).$$

Proof. Checking $(*)$ against Definition 2.3.23, one sees that it is only necessary to show that $(*)$ implies the convergence of point filters. To this end, fix $y \in Y$. Since f is a surjection, there is some $x \in X$ so that $f(x) = y$. Then $[x] \to x$ and $[y] = f([x])$. Thus, $(*)$ implies $[y] \to y$ as desired. QED

Proposition 2.3.29. *Let X be a convergence space, $q : X \to Y$ be a surjection, and Y be given the quotient convergence structure. For any convergence space Ω and continuous function $\varphi : X \to \Omega$ so that for all $x_1, x_2 \in X$ we have $\varphi(x_1) = \varphi_(x_2)$ whenever $q(x_1) = q(x_2)$, there exists a unique continuous map $\varphi^* : Y \to \Omega$ so that*



commutes.

Proof. For each $x \in X$, we may simply define $\varphi^*(q(x)) = \varphi(x)$. In particular, the setup assures us this is a well defined map. It is then clear that φ^* must be continuous since its composition with q is continuous. QED

In Corollary 2.3.20 and Proposition 2.3.26 we saw that the product and convergence sum of topological convergence spaces is topological. With Proposition 2.6.5 and Remark 2.6.10, we will establish the even more powerful result that any initial convergence structure over maps leading into topological convergence spaces is topological. The following result shows just how catastrophically the analogue of this result for final convergence structures fails.

Notation 2.3.30. If X is a set and I some index set recall that the disjoint union of X with itself $|I|$ many times is given by

$$\coprod_{i \in I} X = X \times I.$$

We then define the *collapsing map*

$$\nabla : \coprod_{i \in I} X \to X$$

which is given by $(x, i) \mapsto x$ for each $x \in X$.

Theorem 2.3.31. *Any convergence space is the quotient of a topological convergence space.*

Proof. Let X be a convergence space. Suppose $x \in X$ and $\mathcal{F} \to x$ for some filter $\mathcal{F}$. Define the topology $\tau_{\mathcal{F},x}$ on X by saying U open if and only if $U \notin [x]$ or $U \in \mathcal{F} \cap [x]$. Thus, the neighborhood filter of x is just $\mathcal{F} \cap [x]$ and the neighborhood filter of $z \neq x$ contains $\{z\}$ and is thus $[z]$.

Now, consider the convergence sum over all such $x, \mathcal{F}$ pairs.

$$\Omega = \coprod_{\mathcal{F} \to x} \mathfrak{C}(X, \tau_{\mathcal{F},x}).$$

We will denote elements of Ω as $(z, \tau_{\mathcal{F},x})$ where $z, x \in X$ and $\mathcal{F} \to x$ to distinguish the many copies of X from each other. Let $e_{\mathcal{F},x} : (X, \tau_{\mathcal{F},x}) \to \Omega$ be the usual embedding. The collapsing map ∇ is certainly a surjection. We claim that the convergence structure on X is the quotient convergence structure relative to ∇.

Suppose $x \in X$ and there is a filter $\mathcal{F}$ on X converging to x in the original convergence structure on X. Then, by design, $\mathcal{F} \to x$ in $(X, \tau_{\mathcal{F},x})$. So then $e_{\mathcal{F},x}(\mathcal{F}) \to (x, \tau_{\mathcal{F},x})$. Then we have by continuity of ∇ that $\nabla e_{\mathcal{F},x}(\mathcal{F}) = \mathcal{F} \to x$ in the quotient convergence structure on X.

On the other hand, suppose that $\mathcal{F} \to x$ in the quotient convergence structure on X. One may find finitely many $z_1, ..., z_n \in X$ and filters $\mathcal{G}_i \to z_i$ (in the original convergence structure on X) and filters $\mathcal{F}_i \to (x, \tau_{\mathcal{G}_i, z_i})$ (in Ω) so that

$$\mathcal{F} \supseteq \nabla(\mathcal{F}_1) \cap \nabla(\mathcal{F}_2) \cap \cdots \cap \nabla(\mathcal{F}_n).$$

Let $i \in \{1, ..., n\}$. If $x = z_i$, then $\mathcal{F}_i \supseteq \mathcal{G}_i \cap [x]$. Then $\nabla(\mathcal{F}_i) \to x$ in the original convergence structure on X. Otherwise, suppose that $x \neq z_i$. Then we have that $\mathcal{F}_i = [x]$ and $\nabla(\mathcal{F}_i) \to x$ in the original convergence structure on X. Thus, $\mathcal{F}$ extends the intersection of filters all converging to x and must then itself converge to x in the original convergence structure. QED

2.4 Separation Axioms

In this section, we address the extension of topological separation properties to convergence spaces. We begin by recalling the relevant separation properties. If X is a topological space, X is

T_0 or Kolmogorov	iff	for any distinct $x, y \in X$, there exists open U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$;
T_1 or Fréchet	iff	for all $x \in X$, we have $\{x\}$ closed;
T_2 or Hausdorff	iff	for any distinct $x, y \in X$, there exists open $U \ni x$ and $V \ni y$ such that $U \cap V = \emptyset$.
Functionally Hausdorff	iff	for all distinct $x, y \in X$ there is continuous $f : X \to \mathbb{R}$ so that $f(x) \neq f(y)$.
Regular	iff	For an closed set C and point $x \notin C$ there are open $U \ni x$ and $V \supseteq C$ so that $U \cap V = \emptyset$.

We will give convergence space analogues to these definitions, give net characterizations of these definitions, establish some useful properties, and prove they actually extend the topological notions.

Definition 2.4.1. A convergence space X is called *T_0 or Kolmogorov* when for any $x, y \in X$ at least one of the following holds:

1. there exists a filter $\mathcal{F} \to x$ with $\mathcal{F} \not\to y$;
2. there exists a filter $\mathcal{F} \to y$ with $\mathcal{F} \not\to x$;

Proposition 2.4.2. *A convergence space X is Kolmogorov if and only if for each $x \in X$ at least one of the following holds:*

1. *there exists a net $\alpha \to x$ with $\alpha \not\to y$;*
2. *there exists a net $\alpha \to y$ with $\alpha \not\to x$;*

Proposition 2.4.3. *A topological space X is Kolmogorov if and only if $\mathfrak{C}(X)$ is Kolmogorov.*

Proof. Suppose X is a Kolmogorov topological space. Let $x, y \in X$. Without loss of generality, there is some neighborhood U of x not containing y. We have that $\mathcal{N}_y \to y$, but we cannot have $\mathcal{N}_y \to x$ as this would require $U \in \mathcal{N}(y)$. Thus, $\mathfrak{C}(X)$ is Kolmogorov.

Conversely, suppose $\mathfrak{C}(X)$ is a Kolmogorov. Let $x, y \in X$. Without loss of generality, there is a filter $\mathcal{F} \to x$ with $\mathcal{F} \not\to y$. Thus, $\mathcal{F} \supseteq \mathcal{N}_x$ but $\mathcal{F} \not\supseteq \mathcal{N}_y$. There is thus a neighborhood U of y such that $U \notin \mathcal{F}$. Particularly, $U \notin \mathcal{N}_x$. Thus, U is not a neighborhood of x. QED

Definition 2.4.4. A convergence space X is called T_1 *or Fréchet* when for all $x, y \in X$ we have $[x] \to y$ implies $y = x$.

Proposition 2.4.5. *A convergence space X is Fréchet when for all $x, y \in X$ and nets α with constant value x we have $\alpha \to y$ implies $y = x$.*

Proposition 2.4.6. *A convergence space X is Fréchet if and only if for all $x \in X$ we have $\{x\}$ is closed.*

Proof. Suppose X is a Fréchet convergence space and $x \in X$. We have that $\{x\} \subseteq a(\{x\})$. Suppose $y \in a(\{x\})$. There is then some filter $\mathcal{F} \to y$ with $\{x\} \in \mathcal{F}$. Thus, $[x] \subseteq \mathcal{F}$. But, $[x]$ is an ultrafilter, so $\mathcal{F} = [x]$. Thus, $y = x$. We have $a(\{x\}) = \{x\}$ and $\{x\}$ is closed.

Now, suppose that for all $x \in X$, we have $\{x\}$ is closed. Suppose $x, y \in X$ with $[x] \to y$. We then have that $y \in a(\{x\})$, so $y = x$. Therefore, X is Fréchet. QED

Corollary 2.4.7. *A topological space X is Fréchet if and only if $\mathfrak{C}(X)$ is Fréchet.*

Definition 2.4.8. A convergence space X is called T_2 *or Hausdorff* when if $x, y \in X$ and there is a filter $\mathcal{F} \to x, y$, then it must be that $x = y$.

Proposition 2.4.9. *A convergence space X is Hausdorff if and only if for all $x, y \in W$ with a net $\alpha \to x, y$ it must be that $x = y$.*

Proposition 2.4.10. *A topological space X is Hausdorff if and only if $\mathfrak{C}(X)$ is Hausdorff.*

Proof. Suppose $\mathfrak{C}(X)$ is Hausdorff. Let $x, y \in X$ be distinct points such that for every $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$, we have $U \cap V \neq \emptyset$. We may then consider the filter

$$\mathcal{F} = [\{U \cap V : U \in \mathcal{N}_x \ \wedge \ V \in \mathcal{N}_y\}]$$

which clearly satisfies $\mathcal{F} \supseteq \mathcal{N}(x), \mathcal{N}(y)$. Thus, $\mathcal{F} \to x, y$. We then have $x = y$. Thus, X is Hausdorff in the topological sense.

Now, suppose that X is a Hausdorff topological space. Suppose $x, y \in X$ and $\mathcal{F} \to x, y$. We then have that $\mathcal{N}(x), \mathcal{N}(y) \subseteq \mathcal{F}$. We then have that no two neighborhoods of x and y may have trivial intersection (else $\emptyset \in \mathcal{F}$). Since X is Hausdorff, this means $x = y$. Thus, $\mathfrak{C}(X)$ is Hausdorff. QED

Definition 2.4.11. A convergence space X is called *functionally Hausdorff* when for all distinct $x, y \in X$ there is continuous $f : X \to \mathbb{R}$ so that $f(x) \neq f(y)$.

Proposition 2.4.12. *A convergence space X is functionally Hausdorff if and only if the weak convergence X_σ is Hausdorff.*

Proof. Suppose X_σ is Hausdorff. Suppose $x, y \in X$ are such that for all continuous $f : X \to \mathbb{R}$ we have $f(x) = f(y)$. We then have that $f([x]) \to f(y)$ for all continuous $f : X \to \mathbb{R}$. Therefore, $[x] \to_\sigma y$. Since $[x] \to_\sigma x$ and X_σ Hausdorff, we have that $x = y$.

Now, suppose X is functionally Hausdorff. Suppose $x, y \in X$ and there is a filter $\mathcal{F} \to_\sigma x, y$. Then for all $f : X \to \mathbb{R}$ continuous, we have $f(\mathcal{F}) \to f(x), f(y)$. Since $\mathbb{R}$ is Hausdorff, $f(x) = f(y)$. So, $x = y$ and X_σ Hausdorff. QED

Proposition 2.4.13. *Functionally Hausdorff spaces are Hausdorff.*

Proof. This follows from the observation that convergence implies weak convergence. QED

Definition 2.4.14. A convergence space X is is called *regular* when for every $x \in X$ and filter $\mathcal{F} \to x$ we have $a(\mathcal{F}) \to x$ where

$$a(\mathcal{F}) = [\{a(F) : F \in \mathcal{F}\}].$$

Proposition 2.4.15. *A topological space X is regular if and only if $\mathfrak{C}(X)$ is regular.*

Proof. Suppose X is a regular topological space. Suppose $x \in X$ and that there is a filter $\mathcal{F} \to x$. We then have that $\mathcal{F} \supseteq \mathcal{N}_x$. Let $N \in \mathcal{N}_x$ be open. We then have that $X \smallsetminus N$ is closed. There there are thus open $U \in \mathcal{N}_x$ and $V \supseteq X \smallsetminus N$ which do not intersect. We then have that $U \subseteq X \smallsetminus V$ which is closed. Therefore, $\overline{U} \subseteq X \smallsetminus V$ and $\overline{U} \subseteq N$. Since $\overline{U} \in a(\mathcal{F})$ from $U \in \mathcal{F}$, we have $N \in a(\mathcal{F})$. Thus, $a(\mathcal{F}) \supseteq \mathcal{N}_x$ and $a(\mathcal{F}) \to x$ in $\mathfrak{C}(X)$. Thus, $\mathfrak{C}(X)$ is regular.

Suppose $\mathfrak{C}(X)$ is regular. Let $C \subseteq X$ closed and $x \in X \smallsetminus C$. We then have that

$X \setminus C \in \mathcal{N}_x$. Since $\mathfrak{C}(X)$ is regular, we have $a(\mathcal{N}_x) \supseteq \mathcal{N}_x$. Thus, there is open U with

$$x \in U \subseteq \overline{U} \subseteq X \setminus C.$$

We then have that $x \in U$ and $C \subseteq X \setminus \overline{U}$ which have empty intersection. We conclude that X is regular. QED

Definition 2.4.16. A convergence space X is called functionally regular when for all $x \in X$ and filters $\mathcal{F} \to x$ we have $\overline{\mathcal{F}}^{\sigma} \to x$.

Proposition 2.4.17. *Functionally regular spaces are regular.*

Proof. This follows from the observation that for any filter $\mathcal{F}$ we have $a(\mathcal{F}) \supseteq \overline{\mathcal{F}}^{\sigma}$ which in turn follows from the observation that for any subset F of the ambient space, $\overline{F}^{\sigma} \supseteq a(F)$. QED

Separation often interacts well with subspaces.

Proposition 2.4.18. *If X is a convergence space and $S \subseteq X$ is a subspace, then S is*

1. *Fréchet if X is;*
2. *Hausdorff if X is;*
3. *Functionally Hausdorff if X is;*
4. *Regular if X is;*
5. *Functionally regular if X is.*

Proof. The proofs of (1) and (2) are straightforward, especially if one considers the contrapositive.

For (3), one may separate points of the subspace by continuous functions out of S and then restrict these to S.

For (4), note that if $\mathcal{F}$ is a filter on X, then $[a_S(\mathcal{F})] \supseteq a([\mathcal{F}])$ as filters on X. A similar observation suffices to prove (5). QED

2.5 Compactness

Compactness is an incredibly significant property in the study of topological spaces. This section discusses the extension of this notion to convergence spaces and obtains a painless proof of Tychonoff's theorem. It then covers local compactness.

Definition 2.5.1. A convergence space X is *compact* when every ultrafilter on X converges. A subset K of X is called compact when it is compact once equipped with the subspace convergence structure

Remark 2.5.2. From Theorem 1.4.5,we have that compactness is certainly equivalent to every universal net in X converging.

This succinct definition can be reformulated to be more similar to the statement that in metric spaces a space is compact precisely when it is sequentially compact, that is, each sequence has a convergent subsequence.

Proposition 2.5.3. *If X is a convergence space, the following are equivalent.*

1. *X is compact.*
2. *For every filter $\mathcal{F}$ on X, there is a filter $\mathcal{G} \supseteq \mathcal{F}$ so that $\mathcal{G}$ converges.*
3. *Every net in X has a converging subnet.*

Proof. Suppose X is compact. Let $\mathcal{F}$ be a filter on X. By Theorem 1.4.7, we may find an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$. Since X is compact, $\mathcal{U}$ converges.

Next, suppose that for every filter $\mathcal{F}$ on X, there is a filter $\mathcal{G} \supseteq \mathcal{F}$ so that $\mathcal{G}$ converges. Let α be a net in X. We may find a converging filter $\mathcal{G} \supseteq \mathcal{E}(\alpha)$. We then have that $\eta(\mathcal{G})$ is a converging subnet of α.

Suppose that every net in X has a converging subnet. Thus, since no universal net on X has s proper subnet, each universal subnet of X converges, and X is compact.

We have shown (1) implies (2) implies (3) implies (1) and so have established the desired equivalence. QED

While compactness can be defined as above directly using convergence, it can at times be useful to have a characterization resembling open cover compactness.

Definition 2.5.4. If $X \ni x$ is a convergence space, a *local covering system at x* is a collection $\mathcal{C}$ of subsets of X so that whenever $\mathcal{F} \to x$ in X, we have $\mathcal{F} \cap \mathcal{C} \neq 0$. If $A \subseteq X$, a *covering system* for A in X is a collection $\mathcal{C}$ of subsets of X which is a local covering system at each $x \in A$. We then call a covering system of X in X a *covering system of X*

Remark 2.5.5. Covering systems are a clear generalization of open covers. As such, it is not hard to check that Corollary 2.3.16 and its corollary hold if the open sets involved are replaced with a covering system.

Theorem 2.5.6. *A convergence space X is compact if and only if for each covering system $\mathcal{C}$ of X there exists finite $\mathcal{C}' \subseteq \mathcal{C}$ with $X = \bigcup \mathcal{C}'$.*

Proof. Assume X is a convergence space with covering system $\mathcal{C}$ without finite subset covering X. It is then the case for all $C_1, ..., C_n \in \mathcal{C}$ that

$$\emptyset \neq X \smallsetminus (C_1 \cup C_2 \cup \cdots \cup C_n) = (X \smallsetminus C_1) \cap (X \smallsetminus C_2) \cap \cdots \cap (X \smallsetminus C_n).$$

We may then find an ultrafilter

$$\mathcal{U} \supseteq \{X \smallsetminus C : C \in \mathcal{C}\}$$

There can be no $x \in X$ so that $\mathcal{U} \to x$. As otherwise, there would be $C \in \mathcal{U} \cap \mathcal{C}$. Thus, X is not compact.

For the other direction, assume X is a convergence space so that all covering systems of X have finite subset covering X. Suppose for contradiction there is a non-converging ultrafilter $\mathcal{U}$ on X. We then have that for all converging filters $\mathcal{F}$ on X that $\mathcal{U} \not\supseteq \mathcal{F}$. Thus, for each such $\mathcal{F}$, there is $F_{\mathcal{F}} \in \mathcal{F}$ with $F_{\mathcal{F}} \notin \mathcal{U}$. We then have that

$$\mathcal{C} = \{F_{\mathcal{F}} : \mathcal{F} \text{ a converging filter on } X\}$$

is a covering system with $\mathcal{C} \cap \mathcal{U} = \emptyset$. We find finite $\mathcal{C}' \subseteq \mathcal{C}$ covering X. But since $\mathcal{U}$ is an ultrafilter, we have that $X \smallsetminus C \in \mathcal{U}$ for all $C \in C'$, so

$$\mathcal{U} \ni \bigcap_{C \in C'} (X \smallsetminus C) = X \smallsetminus \bigcup \mathcal{C}' = \emptyset.$$

This contradiction proves that all ultrafilters on X converge and therefore X is compact. QED

Corollary 2.5.7. *A subset K of convergence space X is compact if and only if each covering system of K in X has finite subset covering K.*

Proof. Let X be a convergence space with compact subset K. Let $\mathcal{C}$ be a covering system of K in X. We claim

$$\mathcal{C}^* = \{C \cap K : C \in \mathcal{C}\}$$

is a covering system of K. Let $\alpha \to x$ be a net in K. We have that $\alpha \to x$ in X as well. Thus, there is $C \in \mathcal{C}$ with $\alpha \in_{\text{ev}} C$. Since α takes values only in K, we have that $\alpha \in_{\text{ev}} C \cap K$. Thus, $\mathcal{C}^*$ has finite subset covering K which clearly gives a finite subset of $\mathcal{C}$ covering K.

Assume now that $K \subseteq X$ has the property that all covering systems of K in X have finite subset covering K. Let $\mathcal{C}$ be a covering system of K and define $\mathcal{C}^* = \{C \cup (X \smallsetminus K) : C \in \mathcal{C}\}$. Let $x \in K$ and $\mathcal{F}$ be a filter on X with $\mathcal{F} \to x$ in X. If $(X \smallsetminus K) \in \mathcal{F}$, then there is $C \in \mathcal{C}^*$ with $C \in \mathcal{F}$. Suppose otherwise, that $X \smallsetminus K \notin \mathcal{F}$. Then no $F \in \mathcal{F}$ is contained in $X \smallsetminus K$ and we have $K \cap F \neq \emptyset$. Let

$$\mathcal{F}|_K = [\{F \cap K : F \in \mathcal{F}\}]_K$$

be a filter on K. We observe that its inflation $[\mathcal{F}|_K]_X$ to X satisfies $[\mathcal{F}|_K]_X \supseteq \mathcal{F}$ and so $\mathcal{F}|_K \to x$ in K. There is then $C \in \mathcal{C}$ with $C \in \mathcal{F}|_K$. Thus, there is $F \in \mathcal{F}$ with $C \supseteq F \cap K$ for which it follows that $C \cup (X \smallsetminus K) \supseteq F$ and $C \cup (X \smallsetminus K) \in \mathcal{F} \cap \mathcal{C}^*$. Therefore, $\mathcal{C}^*$ is a covering system of K in X. It therefore, has finite subset covering K. This then gives finite subset of $\mathcal{C}$ covering K. QED

Corollary 2.5.8. *If X is a compact topological space, $\mathfrak{C}(X)$ is compact. If Y is a compact convergence space $\mathfrak{T}(X)$ is compact.*

Proof. Let $\mathcal{C}$ be a covering system for $\mathfrak{C}(X)$. For each $x \in X$, there is $C_x \in \mathcal{C}$ so that $C_x \in \mathcal{N}_x$. Thus, there is an open subset U_x of X with $x \in U_x \subseteq C_x$. Since $\{U_x : x \in X\}$ is an open cover for compact X, it has finite subcover. This witnesses a finite subset of $\mathcal{C}$ covering X. Therefore, $\mathfrak{C}(X)$ is compact.

Any open cover for $\mathfrak{T}(Y)$ is a covering system for Y. Compactness of Y then guarantees a finite subcover. QED

Corollary 2.5.9. *A topological space X is compact if and only if $\mathfrak{C}(X)$ is.*

We now prove some properties of compact spaces which should be familiar from topological or more sequence dependent settings.

Proposition 2.5.10. *Suppose X is a convergence space and $K \subseteq X$ compact. Fix a filter $\mathcal{F}$ on X. If $F \cap K \neq \emptyset$ for all $F \in \mathcal{F}$, and particularly if $K \in \mathcal{F}$, then $\mathcal{F}$ has converging extension.*

Proof. We have that $\mathcal{F}|_K$ is a filter on K. It thus has ultrafilter extension $\mathcal{U}$ which converges in K. Thus, $[\mathcal{U}] \supseteq \mathcal{F}$ converges in X. QED

Proposition 2.5.11. *If X is a compact convergence space and $\mathcal{E}$ is a collection of closed subsets of X so that no finite intersection of the element of $\mathcal{E}$ is empty, then $\bigcap \mathcal{E} \neq \emptyset$.*

Proof. We have that $\mathcal{E}$ generates a filter $[\mathcal{E}]$. Let $\mathcal{U}$ be an ultrafilter extending $[\mathcal{E}]$. We have some $x \in X$ with $\mathcal{U} \to x$. Suppose $E \in \mathcal{E}$, then $E \in \mathcal{U}$. It follows that $x \in a(E)$. But, thus $x \in E$ since E is closed. Therefore, $x \in \bigcap \mathcal{E}$. QED

Proposition 2.5.12. *Let X be a convergence space with closed subset A and compact subset K. The set $A \cap K$ is compact when given its subspace convergence structure.*

Proof. Suppose ω is a universal net in $A \cap K$. By codomain extension, ω is also universal in A, K, and X Since K is compact, $\omega \to x$ for some $x \in K$. Since ω is also a net in A, we have $x \in a(A) = A$. So, $x \in K \cap A$. Each universal net in $A \cap K$ converges, so $A \cap K$ is compact. QED

Corollary 2.5.13. *Closed subsets of compact sets are compact.*

Proposition 2.5.14. *Compact subsets of Hausdorff convergence spaces are closed.*

Proof. Suppose X is a Hausdorff convergence space and $K \subseteq X$ is compact. Suppose that $x \in a(K)$. We then have a net $\alpha \to x$ which is eventually in K. Since K is compact, this net has a subnet converging in K. But this subnet must converge to x as $\alpha \to x$. But limits are unique since X is Hausdorff, so $x \in K$. Therefore, $K = a(K)$ and K is closed. QED

Lemma 2.5.15. *If $f : X \to Y$ is a continuous mapping of convergence spaces and $\mathcal{C}$ is a covering system of Y, then*

$$f^{-1}(\mathcal{C}) = \{f^{-1}(C) : C \in \mathcal{C}\}.$$

is a covering system for X.

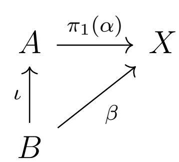
Proof. Suppose $\mathcal{F} \in \Phi(X)$ and $x \in X$ with $\mathcal{F} \to x$. By continuity of f, we have that $f(\mathcal{F}) \to f(x)$. There is then $C \in \mathcal{C}$ with $C \in f(\mathcal{F})$. There is then $F \in \mathcal{F}$ with $C \supseteq f(F)$. Thus, $F \subseteq f^{-1}(C)$ and $f^{-1}(C) \in \mathcal{F}$. Therefore, $\mathcal{F} \cap f^{-1}(\mathcal{C}) \neq \emptyset$ and $f^{-1}(\mathcal{C})$ is a covering system as desired. QED

Proposition 2.5.16. *If X, Y are convergence spaces, X compact, and $f : X \to Y$ a continuous surjection, then Y is compact.*

Proof. Suppose $\mathcal{C}$ is a covering system for Y. By the preceding lemma, we have that $f^{-1}(\mathcal{C})$ is a covering system for X. Since X is compact, there are $C_1, ..., C_n \in \mathcal{C}$ so that $f^{-1}(C_1), ..., f^{-1}(C_n)$ covers X. By surjectivity of f, we have that $C_1, ..., C_n$ covers Y. By Theorem 2.5.6, Y is compact. QED

Proposition 2.5.17. *If X is a compact convergence space, then for any convergence space Y the projection $\pi_2 : X \times Y \to Y$ is closed; that is, whenever $C \subseteq X \times Y$ is closed, $\pi_2(C)$ is also closed.*

Proof. Fix a compact space X, a convergence space Y, and $C \subseteq X \times Y$ closed. Let $\alpha' : A \to \pi_2(C)$ be a net with $\alpha' \to y$ for some $y \in Y$. We produce a net $\alpha : A \to C$ so that $\pi_2(\alpha) = \alpha'$. Since X is compact, $\pi_1(\alpha)$ has a subnet $\beta : B \to X$ converging to some $x \in X$. By Corollary 1.3.7, we may assume that β is a Willard subnet of $\pi_1(\alpha)$. Thus, there is a monotone final map $\iota : B \to A$ so that



commutes. Define $\gamma : B \to X \times Y$ by $\gamma = \alpha \circ \iota$. Since ι is monotone and final, we have that γ is a subnet of α. Therefore, $\pi_2(\gamma) \to y$ since $\pi_2(\gamma)$ is a subnet of $\pi_2(\alpha)$. Further, $\pi_1(\gamma) = \beta$, so $\pi_1(\gamma) \to x$. We conclude that $\gamma \to (x, y)$. Since $\alpha : A \to C$, we may be assured that γ is a net in C. Since C is closed, $(x, y) \in C$. Thus, $y \in \pi_2(C)$. So $\pi_2(C)$ contains its adherence and is closed. QED

Theorem 2.5.18. *If $\{X_i : i \in I\}$ is a family of compact convergence spaces, $X = \prod_{i \in I} X_i$ is compact.*

Proof. Suppose ω is a universal net in X. For each $i \in I$, let $\pi_i : X \to X_i$ be the projection onto the i-th coordinate. By Proposition 1.5.7, we have that each $\pi_i(\omega)$ is universal. Since X_i is compact, $\pi_i(\omega) \to x_i$ for some $i \in I$. By definition of product convergence structure, we have $\omega \to (x_i)_{i \in I}$. Each universal net in X converges, so X is compact. QED

Corollary 2.5.19 (Tychonoff's Theorem)**.** *If $\{X_i : i \in I\}$ is a family of compact topological spaces, $X = \prod_{i \in I} X_i$ is compact.*

Proof. For each $i \in I$, we have by Corollary 2.5.8 that $\mathfrak{C}(X_i)$ is compact. Thus, the product of the $\mathfrak{C}(X_i)$ is compact. Again by Corollary 2.5.8 and Corollary 2.3.19,

$$X = \mathfrak{T}\mathfrak{C}\left(\prod_{i \in I} X_i\right) = \mathfrak{T}\left(\prod_{i \in I} \mathfrak{C}(X)_i\right)$$

is compact. QED

We now give the convergence space analogue of local compactness.

Definition 2.5.20. A convergence space X is called *locally compact* when it is Hausdorff and every converging filter on X contains a compact set.

Remark 2.5.21. We can state this with nets as X is locally compact when it is Hausdorff and every net α converging to $x \in X$ is eventually within a compact set containing x.

Proposition 2.5.22. *Any closed subset of a locally compact topological space is locally compact when endowed with the subspace convergence structure.*

Proof. Let A be a closed subset of a locally compact convergence space X. Give A the subspace convergence structure. Suppose $\mathcal{F}$ is a filter on A and $\mathcal{F} \to a$ for some $a \in A$. Letting $i : A \to X$ be the usual injection, we have that $i(\mathcal{F}) \to a$. We then have that $i(\mathcal{F})$ contains a compact set $K \subseteq X$. There is then some $F \in \mathcal{F}$ so that $K \supseteq F$. We have that $A \cap K \supseteq A \cap F = F$ so that $A \cap K \in \mathcal{F}$. Since A is closed and K compact, we have that $A \cap K$ is compact by Proposition 2.5.12 QED

2.6 Types of Convergence Spaces

In this section we discuss several classes of convergence spaces: topological, pretopological, and Choquet spaces. We have already seen some results pertaining to topological convergence spaces and a functor which turns any convergence space into a topological convergence space. Likewise, each other class of convergence spaces introduced here will have such a functor. Additionally, later sections will introduce yet more classes of convergence spaces and functors transforming general convergence spaces into spaces of the relevant class. Thus, we will first discuss the properties of these functors in abstract.

2.6.1 Modifications

Definition 2.6.1. A *modification of convergence spaces* is a pair (M, μ). Here, M is a functor $M : \mathbf{CONV} \to \mathbf{CONV}$. A convergence space X is called an *M-space* when there is a convergence space Y so that $X \cong M(Y)$. Further, μ is a natural transformation $\mu : \mathrm{id}_{\mathbf{CONV}} \to M$ so that the component of μ at a convergence space X is a homeomorphism if and only if X is an M-space. A modification is called *strict* when each of the components of μ is a bijection.

Example 2.6.2. Suppose $\{X_i : i \in I\}$ is a family of convergence spaces. For any convergence space X, define $M(X)$ to be X with the initial convergence structure relative to

$$\bigcup_{i \in I} C(X, X_i).$$

For each convergence space X define $\mu_X : X \to M(X)$ by $x \mapsto x$. It is not difficult to check that this is a strict modification.

Proposition 2.6.3. *If (M, μ) is a modification, a convergence space X is an M-space if and only if μ_X is a homeomorphism.*

Proof. The result is clear if $X \cong M(X)$ via μ_X. So, suppose X is an M-space. We then have a convergence space Y and homeomorphism $f : X \to M(Y)$. Consider the naturality square

$$\begin{CD} X @>{f}>> M(Y) \\ @V{\mu_X}VV @VV{\mu_{M(X)}}V \\ M(X) @>>{M(f)}> MM(Y) \end{CD}$$

Certainly, $M(Y)$ is an M-space. Thus, $\mu_{M(Y)}$ is a homeomorphism. Thus, $\mu_X = M(f)^{-1}\mu_{M(Y)}f$ is a homeomorphism. QED

Proposition 2.6.4. *Fix a modification (M, μ). If X is a convergence space, Y an M-space, and $f : X \to Y$ is continuous, there is a continuous map $\widehat{f} : M(X) \to Y$ so that*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle \mu_X} \searrow \quad \nearrow {\scriptstyle \widehat{f}} & \\ & M(X) & \end{array}$$

commutes. If (M, μ) is strict, $\widehat{f}$ is unique.

Proof. Certainly, by naturality of μ, we can obtain a function with the desired property by $\widehat{f} = \mu_Y^{-1} \circ M(f)$. If (M, μ) is strict, then given any $\widehat{f}$ making the diagram commute, we may obtain $\widehat{f} = f \circ \mu_X^{-1}$. We note that $\mu_Y^{-1} \circ M(f) = f \circ \mu_X^{-1}$ by naturality of μ. Thus, $\widehat{f}$ is unique. QED

Proposition 2.6.5. *Suppose (M, μ) is a strict modification. If X is a convergence space carrying the initial convergence structure with respect to a family $\{f_i : X \to X_i \mid i \in I\}$ where each X_i is a M-space, then X is an M-space.*

Proof. For each $i \in I$, consider the commutative diagram

$$\begin{CD} X @>f_i>> X_i \\ @A{\mu_X^{-1}}AA @AA{\mu_{X_i}^{-1}}A \\ M(X) @>>{M(f_i)}> M(X_i) \end{CD}$$

Since each X_i is an M-space, we have that each $\mu_{X_i}^{-1}$ is continuous. Thus, for each $i \in I$ we have $f_i \circ \mu_X^{-1}$ is continuous. By the universal property of initial convergence spaces, we have that μ_X^{-1} is continuous. We conclude that X is an M-space. QED

Definition 2.6.6. If (M, μ) is a modification, we define **M-CONV** to be the full subcategory of **CONV** with objects M-spaces and morphisms continuous mappings.

Proposition 2.6.7. *If (M, μ) is a strict modification, M is left adjoint to the inclusion U :* **M-CONV** $\to$ **CONV**.

Proof. Observe that for any convergence space X and M space Y, the diagrams

$$\begin{CD} X @>{\mu_X}>> M(X) \\ @A{\mu_X^{-1}}AA @AA{\mu_{M(X)}^{-1}}A \\ M(X) @>>{M(\mu_x)}> MM(X) \end{CD} \qquad \begin{array}{ccc} M(X) & \xrightarrow{M\mu_X} & MM(X) \\ & \underset{\mathrm{id}_{M(X)}}{\searrow} & \downarrow{\scriptstyle \mu_{M(X)}^{-1}} \\ & & M(X) \end{array}$$

commute (the first by naturality and the second by the first) and trivially

$$\begin{array}{ccc} Y & \xrightarrow{\mu_Y} & M(Y) \\ & \underset{\mathrm{id}_Y}{\searrow} & \downarrow{\scriptstyle \mu_Y^{-1}} \\ & & Y \end{array}$$

commutes. Thus, the adjunction between M and U is witnessed by unit μ and counit μ^{-1}. QED

Corollary 2.6.8. *Modifications are cocontinuous as functors.*

2.6.2 Topological Convergence Spaces

The first major classification of convergence spaces has already been encountered: topological convergence spaces. The associated modification has also been seen before.

Definition 2.6.9. The *topological modification* is the pair (o, τ) where o : **CONV** $\to$ **CONV** is given by $o = \mathfrak{C}\mathfrak{T}$ and τ is the natural transformation with components τ_X introduced in Theorem 2.2.12

Remark 2.6.10. We have from Theorem 2.2.12, Lemma 2.2.13, and Corollary 2.2.16 that (o, τ) is a strict modification of convergence spaces.

The main result of this section is the following classification of topological convergence spaces.

Theorem 2.6.11. *A convergence space X is topological if and only if*

1. *for every $x \in X$, we have $\mathcal{V}_x \to x$;*
2. *for every $S \subseteq X$, we have that $a(S)$ is closed.*

Proof. Suppose X is a topological convergence space. Thus, by Remark 2.2.5, we have that $\mathcal{N}_x = \mathcal{V}_x$ for each $x \in X$; that is, the neighborhood filter from the underlying topological space and the vicinity filter in X coincide. Since $\mathcal{N}_x \to x$, we have $\mathcal{V}_x \to x$. This is (1).

To show that $a(S)$ is closed, we will show $a(S) = \overline{S}$. We already have $a(S) \subseteq \overline{S}$ by Corollary 2.2.24. Now, suppose $x \in \overline{S}$. Then each neighborhood of x intersects S non-trivially. We may thus choose one element from each intersection to construct a net in S converging to x. Thus, $x \in a(S)$ and $a(S) = \overline{S}$ so that $a(S)$ is closed. This is (2).

Now, suppose X is a convergence space satisfying (1) and (2). We will prove $o(X) = X$. Let $x \in X$ and let V be a vicinity of x in X. Every filter $\mathcal{F} \to x$ in X contains V. Thus, no filter converging to x contains $X \setminus V$. Therefore, $x \in X \setminus a(X \setminus V) \subseteq V$. Since $X \setminus a(X \setminus V)$ is open, we have that V is a neighborhood of x. Further, since $\mathcal{V}_x \to x$, we have that $\mathcal{V}_x \supseteq \mathcal{N}_x$ and thus $\mathcal{V}_x = \mathcal{N}_x$. We conclude that a filter in X converges to x if and only if it contains $\mathcal{N}_x$. Therefore, $X = o(X)$ and X is topological. QED

In Proposition 2.5.11, we proved a property of compact spaces which is well known to be equivalent to compactness in topological spaces. We now give a proof of this equivalence in which the role of conditions (1) and (2) in the above are apparent.

Corollary 2.6.12. *A topological space X is compact if and only if whenever $\mathcal{E}$ is a collection of closed subsets of X so that no finite intersection of the element of $\mathcal{E}$ is empty, then $\bigcap \mathcal{E} \neq \emptyset$.*

Proof. The case in which X is compact follows from the like result for convergence spaces given by Proposition 2.5.11.

Suppose whenever $\mathcal{E}$ is a collection of closed subsets of X so that no finite intersection of the element of $\mathcal{E}$ is empty, then $\bigcap \mathcal{E} \neq \emptyset$. Let $\mathcal{U}$ be an ultrafilter on X. Let $\mathcal{E} = \{a(U) : U \in \mathcal{U}\}$. Note that since X is topological and $\mathcal{U}$ is a filter, we have that $\mathcal{E}$ consists of closed sets and has the desired finite intersection property. We thus have that there is some $x \in \bigcap \mathcal{E}$. Thus, for each $U \in \mathcal{U}$, we have some filter $\mathcal{F}_U \to x$ with $U \in \mathcal{F}_U$. Since X is topological, $\mathcal{F}_U \supseteq \mathcal{N}_x$. Thus, for each neighborhood N of x and $U \in \mathcal{U}$, we have $N \cap U \neq \emptyset$. It follows that $X \setminus U \notin \mathcal{U}$. Thus, $N \in \mathcal{U}$ since $\mathcal{U}$ is an ultrafilter. Thus, $\mathcal{U} \supseteq \mathcal{N}_x$. We conclude that $\mathcal{U} \to x$. Therefore, X is compact. QED

Corollary 2.6.13. *If X is a topological convergence space and $S \subseteq X$, then $a(S) = \overline{S}$, the usual topological closure of S in $\mathfrak{T}(X)$.*

Corollary 2.6.14. *If X is a topological space and $S \subseteq X$, the following are equivalent*

1. *S is closed;*

2. *$x \in S$ whenever there is a filter $\mathcal{F} \to x$ with $S \in \mathcal{F}$;*

3. *$x \in S$ whenever there is a net $\alpha \to x$ so that $\alpha \in_{ev} S$.*

2.6.3 Pretopological Spaces

In Theorem 2.6.11, we proved that in order for a convergence space to be topological, the vicinity filter must converge. Such spaces are called pretopological.

Definition 2.6.15. A convergence space X is *pretopological* when $\mathcal{V}_x \to x$ for each $x \in X$.

Remark 2.6.16. In a pretopological space X, a filter $\mathcal{F}$ converges to x if and only if $\mathcal{F} \supseteq \mathcal{V}_x$.

Definition 2.6.17. The *pretopological modification* is the pair (Π, π). The functor Π : **CONV** $\to$ **CONV** sends convergence space X to X equipped with the convergence structure

$$\mathcal{F} \to_\pi x \iff \mathcal{F} \supseteq \mathcal{V}_x$$

and Π has no effect on morphisms. The natural transformation $\pi : \mathrm{id}_{\mathbf{CONV}} \to \Pi$ has component $\pi_X : X \to \Pi(X)$ given by $x \mapsto x$ at convergence space X.

It must of course be proven that the above definition makes sense.

Theorem 2.6.18. *The pretopological modification is a strict modification of convergence spaces.*

Proof. We must verify that

(a) As defined, Π is a functor.

(b) as defined, π is a natural transformation.

(c) The pair (Π, π) is a strict modification.

To check the functoriality of Π, we first verify that if X is a convergence space, then so is $\Pi(X)$. If $x \in X$, then we have that $[x] \to x$ in X. So, $[x] \supseteq \mathcal{V}_x$ and $[x] \to_\pi x$. If filters $\mathcal{F}, \mathcal{G} \to_\pi x$, then $\mathcal{F}, \mathcal{G} \supseteq \mathcal{V}_x$ and $\mathcal{F} \cap \mathcal{G} \supseteq \mathcal{V}_x$ so that $\mathcal{F} \cap \mathcal{G} \to_\pi x$. If $\mathcal{F} \to_\pi x$ and $\mathcal{G} \supseteq \mathcal{F}$, then $\mathcal{G} \supseteq \mathcal{F} \supseteq \mathcal{V}_x$ and $\mathcal{G} \to_\pi x$. Therefore, $\Pi(X)$ is a convergence space.

Next, we must check that if $f : X \to Y$ is a continuous mapping of convergence spaces, then $\Pi(f) : \Pi(X) \to \Pi(Y)$ is continuous. To see this, take $x \in X$ and filter $\mathcal{F} \to_\pi x$. Then, $\mathcal{F} \supseteq \mathcal{V}_x$. By Proposition 2.2.25, we have then that $f(\mathcal{F}) \supseteq \mathcal{V}_{f(x)}$ and $f(\mathcal{F}) \to_\pi f(x)$ as desired for continuity of $\Pi(f)$.

Thus, we have (a), the functoriality of Π. We now consider (b).

Since the necessary naturality diagram is trivial, we need only show that $\pi_X : X \to \Pi(X)$ is continuous for each convergence space X. But this is apparent: any filter $\mathcal{F}$ converging to x in X contains $\mathcal{V}_x$ and so $\mathcal{F} \to_\pi x$. This is exactly the continuity of π_X as required for (b).

Lastly, we consider (c). Strictness is clear. Now, suppose X is a convergence space and π_X is a homeomorphism. By continuity of π_x^{-1}, we have that $\mathcal{V}_x \to x$ since $\mathcal{V}_x \to_\pi x$. This is exactly that $X = \Pi\Pi(X)$. Further, suppose there is some convergence space Y with homeomorphism $f : X \to \Pi(Y)$. Then, we have the commutative naturality diagram

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & \Pi(Y) \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_{\Pi(Y)}} \\ \Pi(X) & \xrightarrow[\Pi(f)]{} & \Pi\Pi(Y) \end{array}$$

It is clear that $\Pi(Y) = \Pi\Pi(Y)$ and so $\pi_{\Pi(Y)}$ is a homeomorphism. Thus, $\pi_X = \Pi(f)^{-1} \circ \pi_{\Pi(Y)} \circ f$ is a homeomorphism. Thus, (Π, π) is a strict modification as desired for (c). QED

Remark 2.6.19. We note from the proof of part (c) that the pretopological spaces defined in Definition 2.6.15 and Π-spaces of Definition 2.6.1 are the same.

Proposition 2.6.20. *If X is a convergence space and $S \subseteq X$, then $a(S) = a_\pi(S)$ where these are the adherences of S in X and $\Pi(X)$ respectively.*

Proof. If $x \in a(S)$, then there is a net α in S with $\alpha \to x$. Since π_X is continuous, $\alpha \to_\pi x$ and $x \in a_\pi(S)$. Thus, $a(S) \subseteq a_\pi(S)$.

Suppose on the other hand that $x \notin a(S)$. Then for every filter $\mathcal{F} \to x$ in X, we have that $F \cap S = \emptyset$ for some $F \in \mathcal{F}$. Otherwise, $\mathcal{F} \cap S \to x$ witnesses that $x \in a(S)$. Thus, $X \setminus S$ is a vicinity of x. therefore, no filter containing S may extend $\mathcal{V}_x$. Thus, $x \notin a_\pi(S)$.

We conclude that $a(S) = a_\pi(S)$ as desired. QED

2.6.4 Choquet Spaces

When studying the convergence of sequences in a metric (or even topological) space, one often makes use of the result[2] that a sequence α failing to converge to a point x is equivalent to there being a subsequence β of α no subsequence of

[2]Though not presented exactly as such, this result is a key idea of Ordman's proof in [Ord66] that almost everywhere convergence is not topological.

which converges to x.

We will see that the convergence space analogue of this result fails to hold in general. First, we will define this analogue and give a few characterizations.

Definition 2.6.21. A convergence space X, is called *Choquet* when a filter $\mathcal{F} \in \Phi(X)$ converges to $x \in X$ whenever every ultrafilter extension of $\mathcal{F}$ converges to x.

Remark 2.6.22. Other sources, *e.g.* [Pat14], refer to these as pseudotopological spaces.

Proposition 2.6.23. *If X is a convergence space, the following are equivalent*

1. *X is Choquet;*
2. *A net α in X converges to $x \in X$ if each of its universal subnets converges to x;*
3. *A filter $\mathcal{F} \in \Phi(X)$ converges to $x \in X$ if for every filter $\mathcal{G} \supseteq \mathcal{F}$ there is a filter $\mathcal{H} \supseteq \mathcal{G}$ so that $\mathcal{H} \to x$;*
4. *A net α in X converges to $x \in X$ if for every subnet β of α there is a subnet γ of β so that $\gamma \to x$.*

Proof. The equivalence of (1) and (2) and the equivalence of (3) and (4) are apparent.

Suppose X is Choquet. Suppose $\mathcal{F} \in \Phi(X)$ and $x \in X$ is such that for every filter $\mathcal{G} \supseteq \mathcal{F}$ there is a filter $\mathcal{H} \supseteq \mathcal{G}$ so that $\mathcal{H} \to x$. If $\mathcal{U} \supseteq \mathcal{F}$ is an ultrafilter, then each of its filter extensions coincide with $\mathcal{U}$, so $\mathcal{U} \to x$. Since X is Choquet, $\mathcal{F} \to x$. This is (3).

Now, suppose (3). Suppose $\mathcal{F} \in \Phi(X)$ and $x \in X$ is such that each ultrafilter $\mathcal{U} \supseteq \mathcal{F}$ converges to x. Suppose there is a filter $\mathcal{G} \supseteq \mathcal{F}$. Extending $\mathcal{G}$ to an ultrafilter $\mathcal{U}$ we have that $\mathcal{U} \to x$. By (3), we have $\mathcal{F} \to x$. Therefore, X is Choquet.

We have now established that (1) and (3) are equivalent. Thus, (1) through (4) are equivalent. QED

Statement (4) in the above is analogous to the contrapositive of that sequential statement at the start of this section. We now show that not all convergence spaces are Choquet.

Example 2.6.24. Consider $\mathbb{R}^2$ with the convergence structure $\mathcal{F} \to x$ when $\mathcal{F}$ converges to x in the usual (topological) convergence structure and $\mathcal{F}$ extends a finite intersection of ultrafilters. It is not hard to check that this is a convergence structure in which the convergence of ultrafilters is exactly the same as in the usual convergence structure on $\mathbb{R}^2$. We claim this is not Choquet.

Let $\mathcal{N}_0$ denote the neighborhood filter of 0 in $\mathbb{R}^2$. Certainly, each ultrafilter extension of $\mathcal{N}_0$ converges to 0. We claim that $\mathcal{N}_0$ does not extend a finite intersection of ultrafilters and thus cannot converge. By Proposition 1.6.4, it suffices to show that $\mathcal{N}_0$ has infinitely many ultrafilter extensions.

Let $N \in \mathbb{N}$ and $v_1, ..., v_N$ be non-parallel unit vectors. Define for each $i = 1, ..., N$

$$\ell_i = \{tv_i : t \in (0,1)\}.$$

and

$$\mathcal{F}_i = \mathcal{N}_0 \cap \ell_i.$$

For each $i = 1, ..., N$ let $\mathcal{U}_i$ be an ultrafilter extension of $\mathcal{F}_i$. Note that $\ell_i \in \mathcal{U}_i$ and $(\mathbb{R}^2 \setminus \ell_j) \in \mathcal{U}_i$ for each $i, j = 1, ..., N$ with $j \neq i$. Thus, the ultrafilters $\mathcal{U}_1, ..., \mathcal{U}_N$ are distinct. As this procedure can be carried out for any $N \in \mathbb{N}$, we have that $\mathcal{N}_0$ has infinitely many ultrafilter extensions. We conclude that $\mathbb{R}^2$ with this convergence structure is not Choquet.

As with topological and pretopological spaces, there is a modification turning any space into a Choquet space.

Lemma 2.6.25. *For each convergence space X, the relation on $\Phi(X) \times X$ given by $\mathcal{F} \to_{ch} x$ exactly when $\mathcal{U} \to x$ for each ultrafilter $\mathcal{U} \supseteq \mathcal{F}$ is a Choquet convergence structure on X.*

Proof. The only property of convergence spaces requiring work is that the intersection of two filters converging to $x \in X$ itself converges to x. This follows from Proposition 1.6.4.

This convergence structure is clearly Choquet since it agrees with the original convergence structure on ultrafilters. QED

Definition 2.6.26. If X is a convergence space, $\mathrm{ch}(X)$ denotes X with the above convergence structure.

Remark 2.6.27. A convergence space X is Choquet if and only if $X = \mathrm{ch}(X)$.

Lemma 2.6.28. *If $f : X \to Y$ is a continuous mapping of convergence spaces, then $ch(f) : ch(X) \to ch(Y)$ defined by $x \mapsto f(x)$ is continuous.*

Proof. Suppose $x \in X$ with filter $\mathcal{F} \to_{\mathrm{ch}} x$. Suppose $\mathcal{U} \supseteq f(\mathcal{F})$ is an ultrafilter. Recall the preimage filter $f^{-1}(\mathcal{U})$ from Proposition 1.5.9.

Suppose $F \in \mathcal{F}$ and $U \in \mathcal{U}$. Since $f(F) \in f(\mathcal{F})$, we have that $f(F) \cap U \neq \emptyset$. Thus, $F \cap f^{-1}(U) \neq \emptyset$. We may thus consider the filter

$$\mathcal{H} = [\{F \cap V : F \in \mathcal{F} \text{ and } V \in f^{-1}(\mathcal{U})\}].$$

Observe that $\mathcal{H} \supseteq \mathcal{F}, f^{-1}(\mathcal{U})$. Let $\mathcal{V} \supseteq \mathcal{H}$ be an ultrafilter. Since $\mathcal{V} \supseteq \mathcal{F}$, we have that $\mathcal{V} \to x$. Since $\mathcal{V} \supseteq f^{-1}(\mathcal{U})$, we have

$$f(\mathcal{V}) \supseteq ff^{-1}(\mathcal{U}) \supseteq \mathcal{U}$$

and $f(\mathcal{V}) = \mathcal{U}$ since $\mathcal{U}$ is an ultrafilter. We then have that $\mathcal{U} \to f(x)$ by continuity of f.

As this holds for all ultrafilters extending $f(\mathcal{F})$, we have that $f(\mathcal{F}) \to_{\text{ch}} f(x)$ and $\text{ch}(f)$ is continuous. QED

Lemma 2.6.29. *For each convergence space X, the map $\chi_X : X \to ch(X)$ given by $x \mapsto x$ is continuous.*

Proof. If $\mathcal{F}$ is a filter on X and $x \in X$ with $\mathcal{F} \to x$, then each ultrafilter extension of $\mathcal{F}$ converges to $x \in X$. Thus, $\mathcal{F} \to_{\text{ch}} x$. QED

Lemma 2.6.30. *If X is a convergence space, then $X \cong ch(Y)$ for some convergence space Y if and only if χ_X is a homeomorphism.*

Proof. If χ_X is a homeomorphism, then $X = \text{ch}(X)$.

Suppose $X \cong \text{ch}(Y)$ for some convergence space Y. We then have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\cong} & \text{ch}(Y) \\ {\scriptstyle \chi_X}\downarrow & & \downarrow{\scriptstyle \chi_{\text{ch}(Y)}} \\ \text{ch}(X) & \xleftarrow[\cong]{} & \text{chch}(Y) \end{array}$$

Since $\text{ch}(Y)$ is Choquet, we have that $\chi_{\text{ch}(Y)}$ is a homeomorphism, from which it follows that χ_X has continuous inverse and is a homeomorphism. QED

The preceding lemmas allow us to prove the following theorem.

Theorem 2.6.31. *The pair (ch, χ) is a strict modification and the ch-spaces of this modification are exactly the Choquet spaces.*

The following is a sometimes useful characterization of convergence in the Choquet modification.

Proposition 2.6.32. *If X is a convergence space and $x \in X$, then a filter $\mathcal{F} \to_{ch} x$ if and only if for every local covering system $\mathcal{C}$ at x in X there are finitely many $C_1, ..., C_n \in \mathcal{C}$ so that $C_1 \cup \cdots \cup C_n \in \mathcal{F}$.*

Proof. Suppose $\mathcal{F} \to_{\text{ch}} x$ and $\mathcal{C}$ is a local covering system at x in X. Suppose that no finite union of elements of $\mathcal{C}$ is contained in $\mathcal{F}$. We may then take an ultrafilter

$$\mathcal{U} \supseteq \{F \cap (X \setminus C) : C \in \mathcal{C},\ F \in \mathcal{F}\}$$

We have that $\mathcal{U} \supseteq \mathcal{F}$ but $\mathcal{U} \cap \mathcal{C} = \emptyset$. This contradicts $\mathcal{F} \to_{\text{ch}} x$, and so $\mathcal{C}$ contains a finite subset whose union is contained in $\mathcal{F}$.

Next, suppose that for every local covering system $\mathcal{C}$ at x in X there are finitely many $C_1, ..., C_n \in \mathcal{C}$ so that $C_1 \cup \cdots \cup C_n \in \mathcal{F}$. Suppose that $\mathcal{U}$ is an ultrafilter on X which does not converge to x. Then, all ultrafilters which do converge to x differ from $\mathcal{U}$ at at least one element. Choose one such element from each converging filter and include them into a local covering system at x. From Corollary 1.4.3 we have $\mathcal{U}$ cannot extend $\mathcal{F}$. Therefore, every ultrafilter extension of $\mathcal{F}$ converges to x and $\mathcal{F} \to_{\text{ch}} x$. QED

We now investigate some of the properties of Choquet spaces, first in relation to the other types of spaces we have seen.

Proposition 2.6.33. *All pretopological spaces are Choquet.*

Proof. Suppose X is a pretopological convergence space. Suppose α is a net in X which does not converge to $x \in X$. It is then the case that $\alpha \notin_{\text{ev}} V$ for some vicinity V of x. We may then find a subnet β of α never within V. This subnet has no subnet converging to x. This is the contrapositive of statement (4) in Proposition 2.6.23. Thus, X is Choquet. QED

Corollary 2.6.34. *All topological spaces are Choquet.*

Corollary 2.6.35. *If X is a convergence space, each of the "identity" maps*

$$X \to ch(X) \to \Pi(X) \to o(X)$$

are continuous.

Proof. We have that $\pi_X : X \to \Pi(X)$ is continuous. Since $\Pi(X)$ is Choquet, we have that the identity $\text{ch}(X) \to \Pi(X)$ is exactly $\text{ch}(\pi_X)$ which is continuous. The other identity is subject to the same reasoning. QED

The following result is familiar from the setting of topological spaces.

Proposition 2.6.36. *Suppose X is a compact Choquet space and Y is Hausdorff. Every continuous bijection $f : X \to Y$ is a homeomorphism.*

Proof. Let α be a net in Y with $\alpha \to y$ for some $y \in Y$. Let ω be a universal subnet of $f^{-1}(\alpha)$. Since X is compact, there is some x so that $\omega \to x$. By continuity of f, we have that $f(\omega) \to f(x)$. Since $f(\omega)$ is a subnet of α, we have $f(\omega) \to y$. Given that Y is Hausdorff, $f(x) = y$. Therefore, $\omega \to f^{-1}(y)$ for each universal subnet of $f^{-1}(\alpha)$ and $f^{-1}(\alpha) \to f^{-1}(y)$ since X is Choquet. We conclude that f^{-1} is continuous and f is a homeomorphism. QED

The requirement that X is Choquet cannot be dropped. To see this, restrict the convergence spaces of Example 2.6.24 to a compact neighborhood K of the origin. Abusively, let K denote this set with its usual convergence structure and K' denote this set with the described non Choquet convergence structure. We have that K' is compact since the convergence of ultrafilters in K' coincides with that in K. The "identity" $K' \to K$ is a continuous bijection between compact Hausdorff spaces which is not a homeomorphism.

Proposition 2.6.37. *Let X and Y be convergence spaces with X Choquet and Y locally compact. If $f : X \to Y$ is a continuous bijection and for each compact $L \subseteq Y$ there exists compact $K \subseteq X$ so that $L \subseteq f(K)$, then f is a homeomorphism.*

Proof. Suppose $f(x) = y \in Y$ and that there is some net $\alpha \to y$ in Y. We wish to show that $f^{-1}(\alpha) \to x$. Since X is Choquet, it suffices to show that every subnet of $f^{-1}(\alpha)$ itself has a subnet converging to x. Let β be a subnet of $f^{-1}(\alpha)$. Since Y is locally compact, we have some some compact $L \subseteq Y$ so that α is eventually in L. It follows that $f^{-1}(\alpha)$ is eventually in $f^{-1}(L)$. By the hypotheses above, we have some $K \subseteq X$ compact so that $f^{-1}(L) \subseteq K$. Therefore, $f^{-1}(\alpha) \in_{\text{ev}} K$. We then have that $\beta \in_{\text{ev}} K$. Now we have that a tail of β is contained in a compact set. Thus, β has a converging subnet. Call this converging subnet γ and its limit z. By the continuity of f, we have that $f(\gamma) \to f(z)$. Since γ is a subnet of $f^{-1}(\alpha)$, we see that $f(\gamma)$ is a subnet of α and thus converges to $f(x)$. But $f(\gamma)$ has a unique limit since Y is Hausdorff. Therefore, $f(x) = f(z)$ and $x = z$ since f is a bijection. We conclude from the fact that X is Choquet that $f^{-1}(\alpha) \to x$. From this we have that f^{-1} is continuous so that f is a homeomorphism. QED

We end this section with a result giving sufficient conditions for the Choquet and pretopological modifications of a space to be topological.

Theorem 2.6.38. *If X is a compact, regular, Hausdorff convergence space, then*

$$ch(X) = \pi(X) = o(X).$$

Proof. To show that $\text{ch}(X) = \Pi(X)$, it is enough to show that the "identity" mapping $\text{ch}(X) \to \Pi(X)$ is a homeomorphism. This map is a continuous bijection, so by Proposition 2.6.36 it suffices to show $\text{ch}(X)$ compact and $\Pi(X)$ Hausdorff. If $\mathcal{U}$ is an ultrafilter on $\text{ch}(X)$, then $\mathcal{U}$ converges in X so it converges in $\text{ch}(X)$ so that $\text{ch}(X)$ is compact. We next show that $\Pi(X)$ is Hausdorff. Suppose there is a filter $\mathcal{F}$ on X with $x, y \in X$ with $\mathcal{F} \to_\pi x, y$. We extend $\mathcal{F}$ to an ultrafilter $\mathcal{U} \to_\pi x, y$. Since X is compact, we have that $\mathcal{U} \to z$ for some $z \in X$. Since X is regular, we have that $a(\mathcal{U}) \to z$. We then see that $a(\mathcal{U}) = a_\pi(\mathcal{U})$ and $a_\pi(\mathcal{U}) \to z$. Suppose $U \in \mathcal{U}$. And consider the filter

$$\mathcal{V}^* = [\{V \cap U : V \in \mathcal{V}_x\}]$$

We have that $U \in \mathcal{V}^*$ and $\mathcal{V}^* \supseteq \mathcal{V}_x$. Therefore, $\mathcal{V}^* \to_\pi x$ and $x \in a_\pi(U)$. It follows that $[x], [y] \supseteq a_\pi(\mathcal{U})$ and $[x], [y] \to z$. Since X is Hausdorff, we have that $x = z = y$. It then follows that $\Pi(X)$ is Hausdorff. At last we have $\Pi(X) = \text{ch}(X)$.

We now prove that $\Pi(X) = o(X)$. We first show that a_π is idempotent. Let $S \subseteq X$. To show that $a(S) = a_\pi(S)$ is closed, it suffices to show that it is compact since X is Hausdorff. To show that $a_\pi(S)$ is compact it suffices to show that $a(A)$ is a compact subset of X since the continuous image of a compact set is compact. We will show this is compact by showing that each of its covering systems has finite subset covering X.

Let $\mathcal{C}_0$ be a covering system for $a(A)$ in X. For each filter $\mathcal{F}$ converging in X to an element of $a(S)$, we have that $a(\mathcal{F})$ converges to the same limit since X is regular. Therefore, $a(\mathcal{F}) \cap \mathcal{C}_0 \neq \emptyset$. There is then some $\mathcal{C}_{\mathcal{F}} \in a(\mathcal{F}) \cap \mathcal{C}_0$. There is then $F_{\mathcal{F}} \in \mathcal{F}$ so that $C_{\mathcal{F}} \supseteq a(F_{\mathcal{F}})$. Letting $\mathcal{C} = \{F_{\mathcal{F}} : \mathcal{F}$ converges to a point in $a(S)$ $\}$, we have another covering system for $a(S)$ each element of which is fully contained in an element of $\mathcal{C}_0$.

We now claim that $\mathcal{C}$ admits finite subset covering S. Suppose otherwise. We may then take an ultrafilter

$$\mathcal{U} \supseteq \{S \smallsetminus C : C \in \mathcal{C}\}$$

Since X is compact, we have that $\mathcal{U}$ converges. Since $S \in \mathcal{U}$, the limit of $\mathcal{U}$ is in $a(S)$. But then $\mathcal{U} \cap \mathcal{C} \neq \emptyset$. This is impossible, so $\mathcal{C}$ has finite subset covering S.

We now may find $C_1, ..., C_n$ so that

$$S \subseteq C_1 \cup C_2 \cup \cdots \cup C_n$$

from which it follows that

$$a(S) \subseteq a(C_1) \cup a(C_2) \cup \cdots \cup a(C_n).$$

We may inflate each of the $C_1, ..., C_n$ to elements of $\mathcal{C}_0$ which thus has a finite subset covering $a(S)$.

We now have that $\Pi(X)$ is pretopological with idempotent closure. Therefore, $\Pi(X)$ is topological and thus $\Pi(X) = o\Pi(X)$. We have that $\mathrm{id} : X \to \Pi(X)$ is continuous. By functoriality of the topological modification, we have that $\mathrm{id} : o(X) \to o\Pi(X) = \Pi(X)$ is continuous. Since $\mathrm{id} : \Pi(X) \to o(X)$ is continuous, we have that $\Pi(X) \cong o(X)$ via the identity map. Immediately, we have $\Pi(X) = o(X)$. QED

Corollary 2.6.39. *Compact, regular, Hausdorff, Choquet spaces are topological.*

2.6.5 Tychonoff Modification

Recall that a topological space X is called *Tychonoff* when for every point $x \in X$ and closed $S \subseteq X \smallsetminus \{x\}$ there is a continuous map $f : X \to [0, 1]$ so that $f(x) = 1$ and $f(S) = \{0\}$. This section details a modification which turns any convergence space into a Tychonoff topological convergence space.

Definition 2.6.40. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$ with its usual topology. For a convergence space X, define an equivalence relation $\sim$ on X by

$$x \sim y \text{ when } f(x) = f(y) \text{ for all } f \in C(X, \mathbb{K})$$

Define $\mathrm{tych}(X) := X/\sim$ and $q : X \to \mathrm{tych}(X)$ to be the quotient map corresponding to $\sim$. For every $f \in C(X, \mathbb{K})$, there is a function $\widetilde{f} : \mathrm{tych}(X) \to \mathbb{K}$ so that

$$\begin{array}{ccc} X & & \\ {\scriptstyle q}\downarrow & \overset{f}{\searrow} & \\ \mathrm{tych}(X) & \xrightarrow[\widetilde{f}]{} & \mathbb{K} \end{array}$$

commutes. The set $\mathrm{tych}(X)$ along with the initial convergence structure over $\{\widetilde{f} : f \in C(X, \mathbb{K})\}$ is called the *Tychonoff* modification of X.

Remark 2.6.41. Since $\mathrm{tych}(X)$ carries the initial convergence structure over $\mathbb{K}$ which is topological, we have by Proposition 2.6.5 that $\mathrm{tych}(X)$ is topological.

There is some concern about the well definition of the Tychonoff modification given that the definition allows the ground field to be either $\mathbb{R}$ or $\mathbb{C}$. We will show that the Tychonoff modification is independent of the choice of ground field.

Proposition 2.6.42. *For a convergence space X we denote by $\mathrm{tych}_{\mathbb{R}}(X)$ the Tychonoff modification of X over ground field $\mathbb{R}$ and denote by $\mathrm{tych}_{\mathbb{C}}(X)$ the Tychonoff modification of X over ground field $\mathbb{C}$, then $\mathrm{tych}_{\mathbb{R}}(X) \cong \mathrm{tych}_{\mathbb{C}}(X)$.*

Proof. Suppose $x, y \in X$ are such that for all $f \in C(X, \mathbb{R})$ we have $f(x) = f(y)$. Let $g \in C(X, \mathbb{C})$. We may write $g = g_1 + g_2 i$ for some $g_1, g_2 \in C(X, \mathbb{R})$ since the projections of $\mathbb{C}$ onto $\mathbb{R}$ are continuous. Thus, $f(x) = g(y)$.

Suppose instead $x, y \in X$ are such that for all $f \in C(X, \mathbb{C})$ we have $f(x) = f(y)$. Let $g \in C(X, \mathbb{R})$. We then have that $e \circ g : X \to \mathbb{C}$ is continuous where $e : \mathbb{R} \to \mathbb{C}$ is the usual inclusion. Thus, $g(x) = g(y)$. We conclude that $\mathrm{tych}_{\mathbb{R}}(X)$ and $\mathrm{tych}_{\mathbb{C}}(X)$ have the same underlying set. Suppose that $f \in C(X, \mathbb{R})$. We then have that $e \circ f : X \to \mathbb{C}$ is continuous. Thus, we have $\widetilde{e \circ f} : \mathrm{tych}_{\mathbb{C}}(X) \to \mathbb{C}$. But the projection of this to $\mathbb{R}$ is just $\widetilde{f}$. Thus, the "identity" map $\mathrm{tych}_{\mathbb{C}}(X) \to \mathrm{tych}_{\mathbb{R}}(X)$ is continuous by the universal property of the initial convergence structure. A similar argument shows that its inverse is continuous. We thus have $\mathrm{tych}_{\mathbb{R}}(X) \cong \mathrm{tych}_{\mathbb{C}}(X)$ where in fact this homeomorphism is an equality. QED

We will next prove that the Tychonoff modification of a convergence space is actually Tychonoff as a topological space. To do so, we will make use of the Stone-Čech compactification (Appendix C.2), the fact that metric spaces, and in particular $\mathbb{R}$ and $\mathbb{C}$, are normal, and the following lemma.

Lemma 2.6.43. *Suppose X is a convergence space and $f \in C(X, \mathbb{K})$. For each $\epsilon > 0$ there is some $g \in C(X, \mathbb{K})$ so that $g(x) = f(x)$ for all $x \in X$ with $|f(x)| \leq \epsilon$ and $|g(x)| = \epsilon$ otherwise.*

Proof. We define $\lambda : X \to \mathbb{K}$ by

$$\lambda(x) = \begin{cases} 1 & |f(x)| \leq \epsilon \\ \epsilon / |f(x)| & \text{else} \end{cases}$$

and claim λ is continuous. We merely check the conditions of Proposition 2.3.15 of which only (4) must be considered in detail. Let $A = \{x \in X : |f(x)| \leq \epsilon\}$ and $B = X \setminus A$.

Since $|f|$ is continuous and $\overline{B}_\epsilon(0)$ closed, no net in A may converge to an element of B.

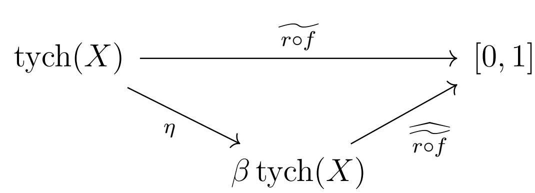
Suppose there is a net in B converging to an element x in A. Again by continuity of $|f|$ it must be that $|f(x)| = \epsilon$ so that $\epsilon/|f(x)| = 1$. We may thus safely use Proposition 2.3.15.

The function $g : X \to \mathbb{K}$ given by $g(x) = \lambda(x)f(x)$ has all of the desired properties. QED

Theorem 2.6.44. *If X is a convergence space,* $\operatorname{tych}(X)$ *is Tychonoff.*

Proof. Fix a convergence space X. From the discussion in Remark 2.6.41, we have that $\operatorname{tych}(X)$ is topological. To show $\operatorname{tych}(X)$ is Tychonoff, it suffices by Proposition C.2.11 to show that $\operatorname{tych}(X)$ is homeomorphic to a subspace of a compact Hausdorff space. Specifically, we will show that the continuous map $\eta : \operatorname{tych}(X) \to \beta \operatorname{tych}(X)$ which maps $\operatorname{tych}(X)$ to its Stone-Čech compactification is an embedding.

We first show that η is an injection. Let $x, y \in X$ be such that $q(x) \neq q(y)$. There is then a continuous map $f : X \to \mathbb{R}$ so that $f(x) \neq f(y)$. Since $\{f(x)\}$ and $\{f(y)\}$ are disjoint closed subsets of $\mathbb{K}$ which is normal, there is continuous $r : \mathbb{K} \to [0,1]$ so that $r(f(x)) = 1$ and $r(f(y)) = 0$. Since $[0,1]$ is compact and Hausdorff, there is by Proposition C.2.7 a unique continuous map $\widehat{\widetilde{r \circ f}} : \beta \operatorname{tych}(X) \to K$ so that



commutes. We thus have that $\widehat{\widetilde{r \circ f}}(\eta(q(x))) = 1$ and $\widehat{\widetilde{r \circ f}}(\eta(q(y))) = 0$ so that $\eta(q(x)) \neq \eta(q(y))$. We conclude that η is an injection.

It remains to show that η has continuous inverse out of its image. Suppose α is a net in X and $x \in X$ so that $\eta(q(\alpha)) \to \eta(q(x))$. To show that η is an embedding, it now suffices to show that $q(\alpha) \to q(x)$ in $\operatorname{tych}(X)$. For this, it must be shown that for any $f : X \to \mathbb{K}$ continuous, we have $\widetilde{f}(q(\alpha)) \to \widetilde{f}(q(x))$.

Let $f : X \to \mathbb{K}$ be continuous. Let U be an open neighborhood of $\widetilde{f}(q(x)) = f(x)$. Let $R > 0$ be such that $f(x) \in B_R(0) \subseteq \mathbb{K}$. By Lemma 2.6.43, there is continuous

$g : X \to \mathbb{K}$ so that

$$\begin{cases} g(x) = f(x) & |f(x)| \leq R \\ |g(x)| = R & \text{else} \end{cases}$$

Thus, setting $K = \overline{B_R(0)}$, we have $g : X \to K$. Since K is compact and Hausdorff there is a unique continuous map $\widehat{\widetilde{g}} : \beta \operatorname{tych}(X) \to K$ so that

$$\begin{array}{ccc} \operatorname{tych}(X) & \xrightarrow{\widetilde{g}} & [0,1] \\ & \searrow^{\eta} \quad \nearrow_{\widehat{\widetilde{g}}} & \\ & \beta \operatorname{tych}(X) & \end{array}$$

commutes. Since $\eta(q(\alpha)) \to \eta(q(x))$, we have that $\widehat{\widetilde{g}}(\eta(q(\alpha))) \to \widehat{\widetilde{g}}(\eta(q(x)))$. By commutativity of the diagram, $\widehat{\widetilde{g}}(\eta(q(x))) = \widetilde{g}(q(x))$. By definition of $\widetilde{g}$, we have $\widetilde{g}(q(x)) = g(x)$. Since x is such that $|f(x)| \leq R$, we have that $g(x) = f(x)$. Thus, $\widehat{\widetilde{g}}(\eta(q(\alpha))) \to f(x)$. In particular, $\widehat{\widetilde{g}}(\eta(q(\alpha)))$ is eventually in $B_R(0) \cap U$ since this is an open set containing $f(x)$. Thus, there is $i_0 \in \operatorname{dom}(\alpha)$ so that for all $i \geq i_0$ we have $\widehat{\widetilde{g}}(\eta(q(\alpha_i))) \in B_R(0) \cap U$. But then for all $i \geq i_0$ we have

$$\widehat{\widetilde{g}}(\eta(q(\alpha_i))) = f(\alpha_i) = \widetilde{f}(q(\alpha_i)).$$

So, $\widetilde{f}(q(\alpha))$ is eventually in U. This holds for all open neighborhoods U of $f(x) = \widetilde{f}(q(x))$, so $\widetilde{f}(q(\alpha)) \to \widetilde{f}(q(x))$. This in turn holds for all $f \in C(X, \mathbb{K})$, so $q(\alpha) \to q(x)$ as desired for η to be an embedding.

We now have that $\operatorname{tych}(X)$ is homeomorphic to a subspace of a compact Hausdorff space. Therefore, $\operatorname{tych}(X)$ is Tychonoff as desired. QED

Corollary 2.6.45. *If X is a functionally Hausdorff convergence space, then the weak convergence X_σ is Tychnoff.*

Proof. Since X is functionally Hausdorff, we have that for any distinct $x, y \in X$ there is $f : X \to \mathbb{R}$ so that $f(x) \neq f(y)$. Thus, $q : X \to \operatorname{tych}(X)$ is a bijection. The initial convergence structure on X over $C(X, \mathbb{K})$ is then identical to that on $\operatorname{tych}(X)$. QED

We now aim to establish that the Tychonoff modification gives rise to a modification in the sense of Definition 2.6.1. The setup for this will consist of a sequence of lemmas.

Lemma 2.6.46. *A convergence space X is a Tychonoff topological space if and only if $X \cong \operatorname{tych}(Y)$ for some convergence space Y.*

Proof. If $X \cong \operatorname{tych}(Y)$ for some convergence space Y, then X is certainly Tychonoff since $\operatorname{tych}(Y)$ is by Theorem 2.6.44.

For the other direction, assume X is Tychonoff. We will prove that the quotient $q : X \to \mathrm{tych}(X)$ is a homeomorphism. We have that q is continuous by the universal property of initial convergence structures. Further, since X is Tychonoff, points may be separated by continuous functions. Therefore, q is a bijection; each of the equivalence classes in $\mathrm{tych}(X)$ is a singleton. To show that q^{-1} is continuous, we will prove the contrapositive: if α is a net in X and $\alpha \not\to x$ for some $x \in X$, then $q(\alpha) \not\to q(x)$.

Fix such a net α and point x. Since X is Tychonoff, it is topological and thus Choquet. We may find a universal subnet ω of α not converging to x. There is then some neighborhood U of x so that $\omega \notin_{\text{ev}} U$. Since ω is universal, $\omega \in_{\text{ev}} X \setminus U$. Since U is open, $X \setminus U$ is closed. Since $x \notin X \setminus U$, we may find a continuous map $f : X \to [0,1]$ so that $f(X \setminus U) = 0$ and $f(x) = 1$. We then see that $f(\omega) \in_{\text{ev}} \{0\}$. Thus, $f(\omega) \not\to f(x) = 1$.

This of course means that $\widetilde{f}(q(\omega)) \not\to f(x)$. Since $\mathrm{tych}(X)$ carries the initial convergence structure, we have that $q(\omega) \not\to q(x)$. Since $q(\omega)$ is a subnet of $q(\alpha)$, we have $q(\alpha) \not\to q(x)$.

From this, we may safely conclude that q^{-1} is continuous and $q : X \to \mathrm{tych}(X)$ is a homeomorphism. QED

Corollary 2.6.47. *A convergence space X is a Tychonoff topological space if and only if $q : X \to \mathrm{tych}(X)$ is a homeomorphism.*

Corollary 2.6.48. *It is the case that $\mathbb{K} \cong \mathrm{tych}(\mathbb{K})$.*

Lemma 2.6.49. *Suppose X and Y are convergence spaces with quotient maps $q_X : X \to \mathrm{tych}(X)$ and $q_Y : Y \to \mathrm{tych}(Y)$. If $f : X \to Y$ is continuous, the map $\mathrm{tych}(f) : \mathrm{tych}(X) \to \mathrm{tych}(Y)$ given by $\mathrm{tych}(f)(q_X(x)) = q_Y(f(x))$ is well defined, continuous, and makes*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_Y} \\ \mathrm{tych}(X) & \xrightarrow[\mathrm{tych}(f)]{} & \mathrm{tych}(Y) \end{array}$$

commute.

Proof. Observe that if it is well defined, the definition of $\mathrm{tych}(f)$ ensures that the diagram commutes.

We first check well definition. Suppose $x, y \in X$ are such that $q_X(x) = q_X(y)$. Suppose $g : Y \to \mathbb{K}$ is continuous. We then have that $g \circ f : X \to \mathbb{K}$ is continuous. Thus, $g \circ f(x) = g \circ f(y)$. This holds for all $g : X \to \mathbb{K}$ continuous, so $q_Y(f(x)) = q_Y(f(y))$. This shows the well definition of $\mathrm{tych}(f)$.

We then show continuity of $\operatorname{tych}(f)$. Let α be a net in X and $x \in X$ so that $q_X(\alpha) \to q_X(x)$. Let $g : Y \to \mathbb{K}$. We have that $g \circ f : X \to \mathbb{K}$ is continuous. We thus have that $\widetilde{g \circ f}(q_X(\alpha)) \to \widetilde{g \circ f}(q_X(x))$. We next observe that

$$\widetilde{g \circ f} = \widetilde{g} \circ \operatorname{tych}(f).$$

We now have $\widetilde{g}(\operatorname{tych}(f)(\alpha)) \to \widetilde{f}(\operatorname{tych}(f)(x))$. This holds for all continuous $g : Y \to \mathbb{K}$, so $\operatorname{tych}(f)(\alpha) \to \operatorname{tych}(f)(x)$. We conclude that $\operatorname{tych}(f)$ is continuous. QED

Remark 2.6.50. The proceeding lemma, along with the observation

$$\operatorname{tych}(\mathbf{id}_X) = \mathbf{id}_{\operatorname{tych}(X)}$$
$$\operatorname{tych}(f \circ g) = \operatorname{tych}(f) \circ \operatorname{tych}(g)$$

for convergence space X and any f and g whose composition is sensible, demonstrates that $\operatorname{tych}(\cdot) : \mathbf{CONV} \to \mathbf{CONV}$ is a functor and $q : \mathbf{id}_{\mathbf{CONV}} \to \operatorname{tych}$ is a natural transformation.

From Lemma 2.6.46 and Lemma 2.6.49, we have that the following definition makes sense.

Definition 2.6.51. The pair (tych, q) is a modification of convergence spaces called the *Tychonoff modification*. The tych-spaces associated to this modification are exactly the Tychonoff topological convergence spaces.

Remark 2.6.52. Unlike the topological, pretopological, and Choquet modifications, the Tychonoff modification is not strict.

Chapter 3

Continuous Convergence

In the last chapter, it was seen that products, coproducts, quotients and other like objects have a natural convergence structure. In this chapter, we will see that the space of continuous functions between two convergence spaces also comes equipped with a canonical convergence structure which, amongst other things, will show that **CONV** is a Cartesian closed category. The results of this chapter are based off those in [BB02] and to a lesser extent [Pat14].

3.1 Remarks on Sets and Topological Spaces

Recall that if A and B are sets, A^B denotes the set of functions $f : B \to A$. Note that for any sets X, Y, A, there is a bijection

$$T_1 : (Y^X)^A \to Y^{A\times X}$$

which we call the *primary transpose*, given by

$$T_1(h)(a, x) = h(a)(x)$$

for all $h \in (Y^X)^A$ and $a \in A$ and $x \in X$. Its inverse, the *secondary transpose*, is given by

$$T_2 : Y^{A\times X} \to (Y^X)^A$$

with

$$T_2(h)(a)(x) = h(a, x)$$

for all $h \in Y^{A\times X}$ and $a \in A$ and $x \in X$. One can further check that this is natural in both A and Y. In the language of category theory, this means that **SET** is Cartesian closed.

The category **TOP** of topological spaces and continuous maps, on the other hand, does not enjoy this property.

Definition 3.1.1. Say that a topological space X is *exponentiable* when for all topological spaces Y, there is a topology on the space of continuous functions $C(X, Y)$ so that the primary transpose

$$T_1 : C(A, C(X, Y)) \to C(A \times X, Y)$$

is a bijection.

In order for **TOP** to be Cartesian closed, it must be that all topological spaces are exponentiable. Unfortunately, this is not the case. Specifically, we have the following definition and result.

Definition 3.1.2. Suppose X is a topological space and $U, V \subseteq X$ are open. We say that $U < V$ when every open cover of V has a finite subcover of U. We say that X is *core-compact* when for all $x \in X$ and for all $V \ni x$ open, there exists $U \ni x$ open so that $U < V$.

Theorem 3.1.3. *A topological space is exponentiable if and only if it is core-compact.*

This is a non-trivial result, and the work required to prove it would take us too far afield. We refer the reader to the elementary treatment of this result in [EH02]. However, it is immediately useful; one can easily show that $\mathbb{Q}$ with its usual topology is not core-compact. Thus **TOP** is not Cartesian closed.

3.2 Continuous Convergence Structure

A major benefit to working with convergence spaces over the usual setting of topological spaces is that the category **CONV** is Cartesian closed. The first thing required to prove this is a convergence structure on the space of continuous functions between a pair of convergence spaces. This will require one supporting definition.

Definition 3.2.1. If X and Y are sets and $A \subseteq Y^X$, define the *evaluation* $\mathrm{ev} : A \times X \to Y$ by $\mathrm{ev}(f, x) = f(x)$. The domain and codomain of evaluation maps should be clear from context, so we refrain from decorating it with specifying notations.

Definition 3.2.2. If X and Y are convergence spaces, define the *continuous convergence structure* on $C(X, Y)$ by

> $\mathscr{F} \to f$ if and only if for all $x \in X$ and $\mathcal{F} \in \Phi(X)$ with $\mathcal{F} \to x$ we have $\mathrm{ev}(\mathscr{F} \times \mathcal{F}) \to f(x)$

for $\mathscr{F}$ a filter on $C(X, Y)$ and $f \in C(X, Y)$.

Notation 3.2.3. The following notational convention will be in place. An undecorated $C(X, Y)$ denotes the set of continuous maps $f : X \to Y$. If the context demands a convergence structure be placed on the function space, then $C(X, Y)$ is always assumed to carry the continuous convergence structure. If the function space at any point carries a different convergence, this will be reflected by decoration; for instance $C_{\mathrm{co}}(X, Y)$ will denote the set $C(X, Y)$ with the compact-open topology. Sometimes, it may be helpful to emphasize that $C(X, Y)$ carries the continuous convergence structure, and in this case we will use the notation $C_{\mathrm{c}}(X, Y)$.

We should verify that the continuous convergence structure is actually a convergence structure.

Proposition 3.2.4. *If X and Y are convergence spaces, the continuous convergence structure is actually a convergence structure.*

Proof. Suppose $f \in C(X,Y)$. Fix $x \in X$ and filter $\mathcal{F} \to x$. One can see that $\mathrm{ev}([f] \times \mathcal{F}) = f(\mathcal{F})$. By continuity of f, we have $\mathcal{F} \to x$. As this holds for all $x \in X$, we have $[f] \to f$ in the continuous convergence structure.

Suppose $\mathscr{F}$ and $\mathscr{G}$ are filters on $C(X,Y)$ so that $\mathscr{F}, \mathscr{G} \to f$ for some $f \in C(X,Y)$. Fix $x \in X$ and filter $\mathcal{F} \to x$. We then see that $(\mathscr{F} \cap \mathscr{G}) \times \mathcal{F} = (\mathscr{F} \times \mathcal{F}) \cap (\mathscr{G} \times \mathcal{F})$. Thus,

$$\mathrm{ev}[(\mathscr{F} \cap \mathscr{G}) \times \mathcal{F}] = \mathrm{ev}[(\mathscr{F} \times \mathcal{F}) \cap (\mathscr{G} \times \mathcal{F})] \to f(x).$$

As this holds for all $x \in X$, we have $\mathscr{F} \cap \mathscr{G} \to f$ in the continuous convergence structure.

Suppose $\mathscr{F}$ and $\mathscr{G}$ are filters in $C(X,Y)$ so that $\mathscr{F} \to f$ for some $f \in C(X,Y)$ and $\mathscr{G} \supseteq \mathscr{F}$. Fix $x \in X$ and filter $\mathcal{F} \to x$. We then have $\mathscr{G} \times \mathcal{F} \supseteq \mathscr{F} \times \mathcal{F}$ so that

$$\mathrm{ev}(\mathscr{G} \times \mathcal{F}) \supseteq \mathrm{ev}(\mathscr{F} \times \mathcal{F}) \to f(x).$$

As this holds for all $x \in X$, we have $\mathscr{G} \to f$ in the continuous convergence structure.

We conclude that the continuous convergence structure is a convergence structure. QED

Evident in the definition of the continuous convergence structure, there is a significant connection between the continuous convergence structure and the evaluation map. This can be formalised as follows.

Proposition 3.2.5. *The evaluation $\mathrm{ev} : C_c(X,Y) \times X \to Y$ is continuous. Further, if $C_*(X,Y)$ is the set of continuous functions with some other convergence structure $\to_*$ so that $\mathrm{ev} : C_*(X,Y) \times X \to Y$ is continuous, then the "identity map" $C_*(X,Y) \to C_c(X,Y)$ is continuous.*

Proof. Suppose $\mathcal{H}$ is a filter in $C(X,Y) \times X$ so that $\mathcal{H} \to (f,x)$ in $C_c(X,Y) \times X$. By continuity of the projections, we have $\pi_1(\mathcal{H}) \to f$ and $\pi_2(\mathcal{H}) \to x$ in $C_c(X,Y)$ and X respectively. By Proposition 1.6.14, we have that $\mathcal{H} \supseteq \pi_1(\mathcal{H}) \times \pi_2(\mathcal{H})$. By definition of continuous convergence, we have $\mathrm{ev}(\pi_1(\mathcal{H} \times \pi_2(\mathcal{H})) \to f(x)$. Thus, $\mathrm{ev}(\mathcal{H}) \to f(x)$. This is exactly what is needed for evaluation to be continuous.

Suppose $\mathscr{F}$ is a filter on $C(X,Y)$ so that $\mathscr{F} \to_* f$ for some $f \in C(X,Y)$. Let $x \in X$ and $\mathcal{F}$ be a filter on X with $\mathcal{F} \to x$. We then have that $\mathscr{F} \times \mathcal{F} \to (f,x)$ in $C_*(X,Y) \times X$. By continuity of evaluation $\mathrm{ev}(\mathscr{F} \times \mathcal{F}) \to f(x)$ in Y. Therefore, $\mathscr{F} \to f$ in $C_c(X,Y)$. This is exactly what is required for the "identity map" $C_*(X,Y) \to C_c(X,Y)$ to be continuous. QED

While the given definition of continuous convergence structure makes use of the abstract power of filters, it is a bit opaque. When translated into the language of nets, it becomes more clear. They key is to use Proposition 1.6.18 to translate between filters and nets in $C(X,Y) \times X$.

Theorem 3.2.6. *Suppose X and Y are convergence spaces. The following are equivalent:*

1. *$C(X,Y)$ has the continuous convergence structure;*

2. *A net Λ in $C(X,Y)$ converges to $f \in C(X,Y)$ if and only if for all $x \in X$ and nets $\alpha \to x$ we have $\mathrm{ev}(\Lambda, \alpha) \to f(x)$.*

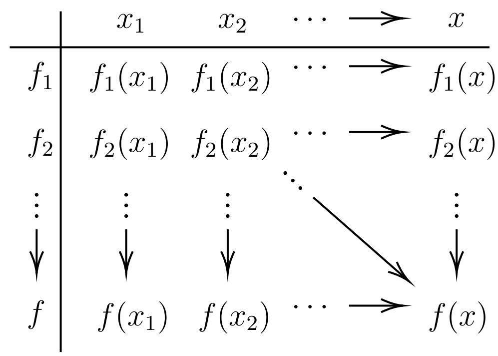
Proof. We begin by assuming that $C(X,Y)$ is equipped with the continuous convergence structure. Suppose Λ is a net in $C(X,Y)$ with $\Lambda \to f$ for some continuous X. Further, suppose α is some net in X converging to x. It is clear that $(\Lambda, \alpha) \to (f, x)$ in $C(X,Y) \times X$. By continuity of the evaluation, $\mathrm{ev}(\Lambda, \alpha) \to f(x)$.

Now, assume that Λ is some net in $C(X,Y)$ so that for some continuous $f : X \to Y$ it is the case that for any $x \in X$ and any net $\alpha \to x$ we have $\mathrm{ev}(\Lambda, \alpha) \to f(x)$. Suppose $x \in X$ and $\mathcal{F} \to x$ is a filter. We have that $\eta(\mathcal{F}) \to x$, from which it follows that $\mathrm{ev}(\Lambda, \eta(\mathcal{F})) \to f(x)$. The net $(\Lambda, \eta(\mathcal{F}))$ has eventuality filter $\mathcal{E}(\Lambda) \times \mathcal{F}$. Thus, $\mathrm{ev}(\Lambda, \eta(\mathcal{F}))$ has eventuality filter $\mathrm{ev}(\mathcal{E}(\Lambda) \times \mathcal{F})$. Therefore, $\mathrm{ev}(\mathcal{E}(\Lambda) \times \mathcal{F}) \to f(x)$. This was true for any $x, \mathcal{F}$ so we have $\mathcal{E}(\Lambda) \to f$ which is exactly what is needed for $\Lambda \to f$. We have established that (1) implies (2).

Now, assume that a net Λ in $C(X,Y)$ converges to $f \in C(X,Y)$ if and only if for all $x \in X$ and nets $\alpha \to x$ we have $\mathrm{ev}(\Lambda, \alpha) \to f(x)$. Suppose $\mathscr{F}$ is some filter on $C(X,Y)$ converging to continuous function f. Fix $x \in X$ and filter $\mathcal{F} \to x$. We then have that $\eta(\mathscr{F}) \to f$ and $\eta(\mathcal{F}) \to x$. We thus have that $\mathrm{ev}(\eta(\mathscr{F}), \eta(\mathcal{F})) \to f(x)$. But we know, $(\eta(\mathscr{F}), \eta(\mathcal{F}))$ has eventuality filter $\mathscr{F} \times \mathcal{F}$. It follows that $\mathrm{ev}(\mathscr{F} \times \mathcal{F}) \to f(x)$.

Now, assume that there is some filter $\mathscr{F}$ on $C(X,Y)$ and some continuous function $f : X \to Y$ so that for all $x \in X$ and filters $\mathcal{F} \to x$ there holds $\mathrm{ev}(\mathscr{F} \times \mathcal{F}) \to f(x)$. Fix any $x \in X$ and net $\alpha \to X$. We see that $\mathrm{ev}(\eta(\mathscr{F}), \alpha) \to f(x)$. As this is true for any x, α we know that $\eta(\mathscr{F}) \to x$. But then $\mathscr{F} \to x$. We have now established that (2) implies (1). QED

This statement of continuous convergence in terms of nets is clearer than the filter version, but can be made clearer still. Consider the case of sequences. Let (x_n) be a sequence in some convergence space which converges to x. Let (f_n) be some sequence of continuous functions converging to f. We have the following chart:



The fact that each f_i and f is continuous means that we may take limits in the horizontal direction. Continuous convergence tells us exactly that we may also take a limit along the "diagonal." Further, it is not hard to show (via the constant net) that we can take vertical limits as well, *i.e.* continuous convergence implies pointwise convergence.

Some evocative notation can also help to make continuous convergence more clear.

Notation 3.2.7. If $\mathscr{F}, \Lambda$ are a filter on and net in $C(X, Y)$ respectively and $\mathcal{F}, \alpha$ are a filter and net in X, define $\mathscr{F}(\mathcal{F}) = \mathrm{ev}(\mathscr{F} \times \mathcal{F})$ and $\Lambda(\alpha) = \mathrm{ev}(\Lambda, \alpha)$. Likewise, if $F \subseteq C(X, Y)$ and $G \subseteq X$, we write $F(G)$ for $\mathrm{ev}(F \times G)$.

We this notation, we have that $\mathscr{F} \to f$ if for all $x \in X$ and filters $\mathcal{F} \to x$ we have $\mathscr{F}(\mathcal{F}) \to f(x)$. A similar statement may be made for nets.

It is useful to know how convergence works in subspaces of a function space equipped with the continuous convergence structure.

Proposition 3.2.8. *Suppose X and Y are convergence spaces and $A(X, Y) \subseteq C(X, Y)$. A filter $\mathscr{F}$ on $A(X, Y)$ converges to $f \in A(X, Y)$ in the subspace convergence structure if and only if $\mathscr{F}(\mathcal{G}) \to f(x)$ for every $x \in X$ and filter $\mathcal{G} \to x$ in X.*

Proof. Let $\iota : A(X, Y) \to C(X, Y)$ be the inclusion mapping. We claim that for any filter $\mathcal{G}$ on X that $\iota(\mathscr{F})(\mathcal{G}) = \mathscr{F}(\mathcal{G})$. Observe

$$\begin{aligned} H \in \mathscr{F}(\mathcal{G}) &\iff \exists F \in \mathscr{F} \;\; \exists G \in \mathcal{G} \;\; (H \supseteq F(G)) \\ &\iff \exists F' \in \iota(\mathscr{F}) \;\; \exists G \in \mathcal{G} \;\; (H \supseteq F'(G)) \\ &\iff H \in \iota(\mathscr{F})(\mathcal{G}) \end{aligned}$$

as desired to show set equality.

Since $A(X, Y)$ carries the subspace convergence structure, $\mathscr{F} \to f$ if and only if $\iota(\mathscr{F}) \to f$. The desired result then follows by the definition of the continuous convergence structure. QED

Now that we have the basics of continuous convergence in place, we consider how these interact with the transpositions of Section 3.1.

Proposition 3.2.9. *If A,X, and Y are convergence spaces and $h : A \to C(X,Y)$ is continuous, then the primary transpose $T_1(h) : A \times X \to Y$ is continuous.*

Proof. This is immediate once one notices $T_1(h) = \mathrm{ev} \circ (h \times \mathrm{id}_X)$. QED

Proposition 3.2.10. *If A, X, and Y are convergence spaces and $h : A \times X \to Y$ is continuous, then $T_2(h) : A \to C(X,Y)$ and $T_2(h)$ is continuous.*

Proof. Suppose $h : A \times X \to Y$. We claim that for each $a \in A$, the map $T_2(h)(a)$ is continuous. If β is any net in X converging to x, we have $T_2(h)(a)(\beta) \sim h(\alpha, \beta)$ where α is the constant net at a. Then $(\alpha, \beta) \to (a, x)$ and $h(\alpha, \beta) \to h(a, x)$ by continuity of h. So, $T_2(h)(a)(\beta) \to T_2(h)(a)(x) = h(a, x)$ as desired for continuity of $T_2(h)(a)$.

Now, suppose $\mathscr{F}$ is a filter on A with $\mathscr{F} \to a$ for some $a \in A$. Fix $x \in X$ and filter $\mathcal{F} \to x$. We see that $\mathscr{F} \times \mathcal{F} \to (a, x)$. By continuity of h we obtain $h(\mathscr{F} \times \mathcal{F}) \to h(a, x) = T_2(h)(a)(x)$. We next claim that $h(\mathscr{F} \times \mathcal{F}) = \mathrm{ev}(T_2(h)(\mathscr{F}) \times \mathcal{F})$. which may be proven by a simple double containment. We have that $\mathrm{ev}(T_2(h)(\mathscr{F}) \times \mathcal{F}) \to T_2(h)(a)(x)$. Moreover, as this holds for any $x \in X$ and filter $\mathcal{F} \to x$, we may conclude that $T_2(h)(\mathscr{F}) \to T_2(h)(a)$. This is exactly what it means for $T_2(h)$ to be continuous. QED

Since T_1 and T_2 invert each other, we obtain

Corollary 3.2.11. *Fix convergence spaces A, X, and Y. A function $h : A \to C(X,Y)$ is continuous if and only if $T_1(h) : A \times X \to Y$ is continuous. A function $g : A \times X \to Y$ is continuous if and only if $T_2(g) : A \to C(X,Y)$ and $T_2(g)$ is continuous.*

This result is exceedingly useful. Trying to prove a map $h : A \to C(X,Y)$ is continuous directly from Definition 3.2.2 or Theorem 3.2.6 is a tedious process involving the consideration of several different filters or nets spread across spaces A, X, and Y. Testing for continuity with transposition, on the other hand, often allows one to conclude continuity simply from the commutativity of a diagram or straightforward computation. This technique is used repeatedly in the proof of the following theorem.

Theorem 3.2.12. *If A, X, and Y are convergence spaces, then*

$$T_1 : C(A, C(X,Y)) \to C(A \times X, Y)$$

is a homeomorphism.

Proof. From Corollary 3.2.11, we have that T_1 is a bijection with inverse T_2. It remains to show that T_1 and T_2 are continuous.

To show that T_1 is continuous, it suffices to show that

$$T_1(T_1) : C(A, C(X,Y)) \times A \times X \to Y$$

is continuous. This follows from commutativity of

$$\begin{array}{ccc} C(A, C(X,Y)) \times A \times X & & \\ \downarrow{\scriptstyle \mathrm{ev} \times \mathrm{id}_X} & \searrow^{T_1(T_1)} & \\ C(X,Y) \times X & \xrightarrow[\mathrm{ev}]{} & Y \end{array}$$

To show that T_2 is continuous, it is enough to show that

$$T_1^2(T_2) : C(A \times X, Y) \times A \times X \to Y$$

is continuous. Let $f \in C(A \times X, Y)$ and $a \in A$ and $x \in X$. We then have that

$$\begin{aligned} T_1^2(T_2)(f, a, x) &= (T_1(T_2))(f, a)(x) \\ &= ((T_2(f))(a))(x) \\ &= f(a, x) \end{aligned}$$

Thus, $T_1^2(T_2)$ is just an evaluation[3] which is certainly continuous. We conclude the desired homeomorphism. QED

An important consequence of this theorem is that **CONV** is Cartesian closed.

We now prove some properties related to the continuous convergence structure.

Proposition 3.2.13. *If X and Y are convergence spaces and X non-empty, then Y is homeomorphic to a subspace of $C(X, Y)$.*

Proof. For each $y \in Y$, define $y^* \in C(X, Y)$ to be the constant map with value y. Let $Y^* = \{y^* \in C(X, Y) : y \in Y\}$. We claim that $h : Y \to Y^*$ given by $h(y) = y^*$ is a homeomorphism. It is a bijection, so we must only show that it is continuous with continuous inverse.

Observe that $T_1(h) : Y \times X \to Y$ is simply the projection onto the Y coordinate. This is continuous, so h is continuous.

We must now show that h^{-1} is continuous. Suppose that Λ is a net in Y so that $\Lambda^* \to y^*$ for some $y \in Y$. Let x be a constant net in X. We have that $\Lambda^*(x) \to y^*(x)$. This is exactly that $\Lambda \to y$. So, h^{-1} is invertible and we conclude the desired homeomorphism. QED

Corollary 3.2.14. *If X and Y are convergence spaces and X non-empty, then $C(X, Y)$ is*

1. *Fréchet if and only if Y is;*
2. *Hausdorff if and only if Y is;*
3. *Choquet if and only if Y is.*

[3] Note that this is not precisely true. Both this part of the proof and the preceding part hide homeomorphisms mediating the "associativity" of the product of convergence spaces. It is more proper to say $T_1^2(T_2)$ is the composition of a homeomorphism and evaluation.

Proof. (1) If $C(X,Y)$ is Fréchet, then Y is by Proposition 2.4.18 since Y is homeomorphic to a subspace of $C(X,Y)$. Suppose $C(X,Y)$ is not Fréchet. There is then a constant net Λ in $C(X,Y)$ with value f so that $\Lambda \to g$ with $g \neq f$. We may then choose $x \in X$, so that $\Lambda(x) \to g(x)$ with $f(x) \neq g(x)$. But $\Lambda(x)$ is merely a constant net with value $f(x)$. Thus, Y is not Fréchet.

(2) follows from analogous reasoning.

(3) If $C(X,Y)$ is Choquet, then Y is since Y is homeomorphic to a subspace of $C(X,Y)$.

Suppose now that Y is Choquet. To show that $C(X,Y)$ is Choquet, it suffices to show that it is equal to its Choquet modification. By Proposition 2.6.32, it suffices to show that for any $f \in C(X,Y)$, if a filter $\mathscr{F}$ is such that for all local covering systems of f there is a finite subcollection whose union is contained in $\mathscr{F}$, then $\mathscr{F} \to f$.

Fix f and $\mathscr{F}$ with these properties. Let $x \in X$ and $\mathcal{F} \to x$ for some filter $\mathcal{F}$ on X. Suppose $\mathcal{C}$ is a local covering system at $f(x)$ in Y. For each $C \in \mathcal{C}$ and $F \in \mathcal{F}$, define

$$\langle C, F\rangle = \{g \in C(X,Y) : C \supseteq g(F)\}$$

and then

$$\mathscr{C} = \{\langle C, F\rangle : C \in \mathcal{C},\ F \in \mathcal{F}\}.$$

One may check that $\mathscr{C}$ is a local covering system at f is $C(X,Y)$. There are then finitely many $C_1, ..., C_n \in \mathcal{C}$ and $F_1, ..., F_n \in \mathcal{F}$ so that

$$\langle C_1, F_1\rangle \cup \cdots \cup \langle C_n, F_n\rangle \in \mathscr{F}.$$

We then have that

$$C_1 \cup \cdots \cup C_n \supseteq (\langle C_1, F_1\rangle \cup \cdots \cup \langle C_n, F_n\rangle)(F_1 \cap \cdots \cap F_n) \in \mathscr{F}(\mathcal{F}).$$

We conclude from this that any covering system of $f(x)$ in Y contains a finite subset whose union lies in $\mathscr{F}(\mathcal{F})$. Thus, by Proposition 2.6.32 and since Y is Choquet, we have $\mathscr{F}(\mathcal{F}) \to f(x)$. This is the case for all $x \in X$ and filters $\mathcal{F} \to x$. So, $\mathscr{F} \to f$. This is the desired result, and we conclude that $C(X,Y)$ is Choquet. QED

A similar result holds for functional Hausdorffness and regularity, but requires a lemma.

Lemma 3.2.15. *Fix convergence spaces X and Y. If Λ is a net in $C(X,Y)$ and $\Lambda \to_\sigma f$, then $\Lambda(x) \to_\sigma f(x)$ for all $x \in X$.*

Proof. We must show that for any $g \in C(Y)$ that $g \circ \Lambda(x) \to g(f(x))$. Define $H : C(X,Y) \to \mathbb{K}$ by $H(h) = g \circ h(x)$ for all continuous $h : X \to Y$. We claim that H is continuous. This follows from commutativity of

$$\begin{array}{ccc} C(X,Y) & \xrightarrow{\quad H \quad} & \mathbb{K} \\ {\scriptstyle \mathrm{id}}\downarrow & & \uparrow{\scriptstyle \mathrm{ev}} \\ \{g\} \times C(X,Y) \times \{x\} & \xrightarrow[\circ \times \mathrm{id}]{} & C(X) \times \{x\} \end{array}$$

Thus, $H \in C_c C_c(X,Y)$. We then have that $H(\Lambda) \to H(f) = g(f(x))$ and that $H(\Lambda) = g \circ \Lambda(x)$. This is the desired convergence. We thus have that $\Lambda(x) \to_\sigma f(x)$. QED

Proposition 3.2.16. *If X and Y are convergence spaces, then $C_c(X,Y)$ is functionally regular or functionally Hausdorff if and only if Y is.*

Proof. One direction is trivial since by Proposition 3.2.13 we may embed Y into $C_c(X,Y)$.

Assume Y is functionally regular. Suppose we have a filter $\mathscr{F} \to f$ in $C_c(X,Y)$. We wish to show that $\overline{\mathscr{F}}^\sigma \to f$. Let $\mathcal{G} \to x$ in X. We have that $\mathscr{F}(\mathcal{G}) \to f(x)$. Since Y is functionally regular, we have $\overline{\mathscr{F}(\mathcal{G})}^\sigma \to f(x)$. We claim that $\overline{\mathscr{F}(\mathcal{G})}^\sigma \subseteq \overline{\mathscr{F}}^\sigma(\mathcal{G})$, so that $\overline{\mathscr{F}}^\sigma(\mathcal{G}) \to f(x)$. To verify this claim, let $U \in \overline{\mathscr{F}(\mathcal{G})}^\sigma$. There is then $F \in \mathscr{F}$ and $G \in \mathcal{G}$ so that $U \supseteq \overline{F(G)}^\sigma$. Let $y \in \overline{F}^\sigma(G)$. There is then $x \in G$ and $f \in \overline{F}^\sigma$ so that $f(x) = y$. We then have a net Λ in F so that $\Lambda \to_\sigma f$. We then have that $\Lambda(x) \to_\sigma f(x)$ by Lemma 3.2.15. Thus, $f(x) = y \in \overline{F(G)}^\sigma$. Then one has that $\overline{F}^\sigma(G) \subseteq \overline{F(G)}^\sigma$ from which the claim follows. It then follows that $\overline{\mathscr{F}}^\sigma \to f$ so that $C_c(X,Y)$ is functionally regular.

Now suppose that Y is functionally Hausdorff. Suppose that we have a net $\Lambda \to_\sigma f, g$ in $C(X,Y)$. For each $x \in X$, we have that $\Lambda(x) \to_\sigma f(x), g(x)$. Since Y is functionally Hausdorff, we have that $f(x) = g(x)$ so that $f = g$ and $C_c(X,Y)$ is functionally Hausdorff. QED

Corollary 3.2.14 and Proposition 3.2.16 give us an important class of functionally regular and functionally Hausdorff Choquet spaces.

Definition 3.2.17. Recall that $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$. We denote by $C(X)$ the convergence space $C(X, \mathbb{K})$. This convergence space, called the *paradual* of X, is Hausdorff and Choquet since $\mathbb{K}$ is. Let $C^b(X)$ denote the subset of $C(X)$ consisting of bounded functions. Unless otherwise indicated, this carries the subspace convergence structure from $C_c(X)$.

As we will show, the bounded functions are dense in the paradual.

Proposition 3.2.18. *If X is a convergence space, then the set $C^b(X)$ of bounded continuous function $f : X \to \mathbb{K}$ is dense in $C(X)$.*

Proof. Let $f \in C(X)$. Consider $\mathbb{R}^+$ as a directed set under its usual order. Consider the net $\Lambda : \mathbb{R}^+ \to C(X)$ so that for each $r \in \mathbb{R}^+$ and $x \in X$ we have $\Lambda_r(x) = f(x)$ for all $x \in X$ with $|f(x)| \leq r$ and $|\Lambda_r(x)| = r$ otherwise. Such continuous functions exist by Lemma 2.6.43. We claim that $\Lambda \to f$.

Suppose $x \in X$ with a net $\alpha \to x \in X$ with domain D. Let $\epsilon > 0$ By continuity of f, we have $f(\alpha) \to x$. Without loss of generality, let R bound $|f(\alpha)|$ above (if $|f(\alpha)|$ is unbounded, we restrict ourselves to a bounded tail and invoke Corollary 2.1.9). Let $d_0 \in D$ so that for all $d \geq d_0$ we have $|f(\alpha_d) - f(x)| < \epsilon$. Observe that if $r \geq R$ then for all $d \in D$ we have $\Lambda_r(\alpha_d) = f(\alpha_d)$. So, when $(r, d) \geq (R, d_0)$ we have $|\Lambda_r(\alpha_d) - f(x)| < \epsilon$. Therefore, $\Lambda(\alpha) \to f(x)$ as desired for $\Lambda \to f$. QED

Continuous maps between convergence spaces induce continuous maps between function spaces.

Definition 3.2.19. Fix a convergence space Z. For any convergence spaces X, Y and continuous $f : X \to Y$, define $f^* : C(Y, Z) \to C(X, Z)$ by $f^*(h) = h \circ f$. Define $f_* : C(Z, X) \to C(Z, Y)$ by $f_*(h) = f \circ h$.

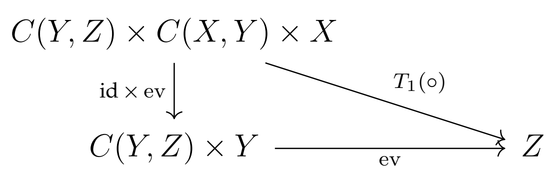
The continuity of these maps is a consequence of the following lemma.

Lemma 3.2.20. *If X, Y, Z are convergence spaces, the composition map*

$$\circ : C(Y, Z) \times C(X, Y) \to C(X, Z)$$

is continuous.

Proof. We check that



commutes so that $\circ$ is continuous by Corollary 3.2.11. QED

Corollary 3.2.21. *If $f : X \to Y$ is a continuous mapping of convergence space, then for any convergence space Z, the maps $f^* : C(Y, Z) \to C(X, Y)$ and $f_* : C(Z, X) \to C(Z, Y)$ are continuous.*

Remark 3.2.22. It is the straightforward to check that for all convergence spaces X, Y, W and $f : X \to Y$ and $g : Y \to W$ continuous that

1. $\text{id}_X^* = \text{id}_{C(X,Z)}$;
2. $\text{id}_{X*} = \text{id}_{C(Z,X)}$;
3. $(g \circ f)^* = f^* \circ g^*$;
4. $(g \circ f)_* = g_* \circ f_*$.

From this, we have that the assignments

$$X \mapsto C_c(X, Z)$$

and

$$f \mapsto f^*$$

for convergence spaces X and continuous f is a contravariant functor **CONV** $\to$ **CONV**. Likewise, the assignment

$$X \mapsto C_c(Z, X)$$

and

$$f \mapsto f_*$$

for convergence spaces X and continuous f is a covariant functor **CONV** $\to$ **CONV**. Moreover, the action of these functors on morphisms is continuous.

Proposition 3.2.23. *Fix a convergence space Z. For any convergence spaces X and Y, the maps*

$$(\cdot)^* : C_c(X, Y) \to C_c(C_c(Y, Z), C_c(X, Z))$$

and

$$(\cdot)_* : C_c(X, Y) \to C_c(C_c(Z, X), C_c(Z, Y))$$

are continuous.

Proof. Continuity follows from the fact that the primary transpose of each map is merely composition which is continuous. QED

3.3 The Compact-Open Topology

Under certain circumstances, the continuous convergence structure is actually a topological convergence structure. This section considers such a case.

Definition 3.3.1. If X and Y are sets and $K \subseteq X$ and $U \subseteq Y$, define

$$T(K, U) = \{f \in Y^X : f(K) \subseteq U\}.$$

If X and Y are convergence spaces, then we define the *compact-open* topology on $C(X, Y)$ to be that generated by subbasis

$$\mathscr{B}_{\text{co}} = \{T(K, U) : K \subseteq X \text{ compact and } U \subseteq Y \text{ open}\}.$$

We denote $C(X, Y)$ with this convergence structure by $C_{\text{co}}(X, Y)$.

The topology induced by the continuous convergence structure is finer than the compact open topology.

Proposition 3.3.2. *For any convergence spaces X and Y, the "identity" map $\mathfrak{T}C_c(X,Y) \to C_{\text{co}}(X,Y)$ is continuous.*

Proof. It suffices to show that each subbasic open set of $C_{\text{co}}(X,Y)$ is open in $\mathfrak{T}C(X,Y)$. Fix $K \subseteq X$ compact and $U \subseteq Y$ open. Suppose $\mathscr{F}$ is a filter on $C(X,Y)$ so that $T(K,U) \notin \mathscr{F}$. Then for each $F \in \mathscr{F}$ we have $F \not\subseteq T(K,U)$. Thus, we may define the non-empty set

$$H_F = \{x \in K : \exists f \in F(f(x) \notin U)\}.$$

Observe that if $F_1, F_2 \in \mathscr{F}$, then $H_{F_1 \cap F_2} \subseteq H_{F_1} \cap H_{F_2}$. We may thus safely define a filter $\mathcal{H} \in \Phi(X)$ by

$$\mathcal{H} = [\{H_F : F \in \mathscr{F}\}].$$

We observe that $K \in \mathcal{H}$, so by compactness of K and Proposition 2.5.10 we have that there is some ultrafilter $\mathcal{V} \supseteq \mathcal{H}$ so that $\mathcal{V} \to x$ for some $x \in K$. We claim that $U \notin \mathscr{F}(\mathcal{V})$. Otherwise, there is $F \in \mathscr{F}$ and $V \in \mathcal{V}$ so that $F(V) \subseteq U$. But then $V \cap H_F = \emptyset$, a contradiction.

Thus, for each $f_0 \in T(K,U)$ we have that $\mathscr{F}(\mathcal{V}) \not\to f_0(x)$ since U is an open set containing $f_0(x)$. Therefore, $\mathscr{F} \not\to f_0$. By contraposition, we have that $T(K,U)$ is a vicinity of each of its elements, meaning that it is open.

We now have that the subbasic open sets of $C_{\text{co}}(X,Y)$ are open in $\mathfrak{T}C(X,Y)$. We conclude that the "identity" map $\mathfrak{T}C(X,Y) \to C_{\text{co}}(X,Y)$ is continuous. QED

Corollary 3.3.3. *For any convergence spaces X and Y, the "identity" map $C_c(X,Y) \to \mathfrak{C}C_{\text{co}}(X,Y)$ is continuous.*

Proposition 3.3.4. *If X is locally compact and Y is regular and topological, then $C_c(X,Y) = \mathfrak{C}C_{\text{co}}(X,Y)$.*

Proof. It suffices to show that convergence in $C_{\text{co}}(X,Y)$ implies convergence in $C_c(X,Y)$. Suppose $f \in C(X,Y)$ with filter $\mathscr{F} \to_{\text{co}} f$. Fix $x \in X$, an open neighborhood U of $f(x)$, and a filter $\mathcal{G} \to x$. By continuity of f, we have that $f(\mathcal{G}) \to f(x)$ and $\overline{f(\mathcal{G})} \to f(x)$ by regularity and topologicity of Y. Thus, $U \in \overline{f(\mathcal{G})}$ and there is $G \in \mathcal{G}$ so that $U \supseteq \overline{f(G)}$. Further, by continuity of f, we have that $U \supseteq f(\overline{G})$. Note that $\overline{G} \in \mathcal{G}$. Since X is locally compact, there is compact $K' \subseteq X$ so that $x \in K' \in \mathcal{G}$. We then have that $K := \overline{G} \cap K' \in \mathcal{G}$. We have that $f(K) \subseteq U$, so $f \in T(K,U)$ which is open in $C_{\text{co}}(X,Y)$. Since $\mathscr{F} \to_{\text{co}} f$, we have that $T(K,U) \in \mathscr{F}$. Thus, $U \supseteq T(K,U)(K) \in \mathscr{F}(\mathcal{G})$. This holds for all open $U \ni f(x)$, so $\mathscr{F}(\mathcal{G}) \to f(x)$. Indeed, this is true for any $x \in X$ and filter $\mathcal{G} \to x$. Therefore, $\mathscr{F} \to f$ in the continuous convergence structure. This is the desired result. QED

Corollary 3.3.5. *Whenever X is a locally compact convergence space, its paradual is topological with the compact-open topology.*

Theorem 3.3.6. *If X is a compact topological convergence space, then $C(X) = C_{co}(X)$ is a Banach space with supremum norm.*

Proof. Define $||\cdot|| : C_c(X) \to \mathbb{K}$ by $||f|| = \sup\{|f(x)| : x \in X\}$ for all continuous $f : X \to \mathbb{K}$.

We first argue that this is well defined, that is each $f \in C_c(X)$ is bounded. Suppose for the sake of contradiction there is an unbounded $f \in C_c(X)$. There is then some net $\alpha : \mathbb{N} \to X$ so that for all $n \in \mathbb{N}$ we have $|f(\alpha_n)| > n$. Since X is compact, we have that some subnet β of α so that $\beta \to x$ for some $x \in X$. But since $f(\alpha) \in_{\text{ev}} \mathbb{K} \setminus n\mathbb{D}$ for all $n \in \mathbb{N}$, we have that $f(\beta) \in_{\text{ev}} \mathbb{K} \setminus n\mathbb{D}$ as well. So, $f(\beta)$ cannot converge. This contradicts the continuity of f. We conclude that all $f \in C_c(X)$ are bounded.

We next argue that the topology induced by this norm is the compact-open topology. Let $U \subseteq \mathbb{K}$ be open and $K \subseteq X$ compact. Let $f \in T(K, U)$. For every $k \in K$, we may find $r_k > 0$ so that $B_{r_k}(f(k)) \subseteq U$. The collection $\{B_{r_k/2}(f(k)) : k \in K\}$ is an open cover for $f(K)$ which is compact. We may thus find $k_1, ...k_n$ so that $\{B_{r_{k_i}/2}(f(k_i)) : i = 1, ..., n\}$ is a finite subcover. Let $\epsilon = \min\{r_{k_i} : i = 1, ..., n\}/2$. We claim that $B_\epsilon(f) \subseteq T(K, U)$. Let $g \in B_\epsilon(f)$ and $x \in K$. There is some $k_i = k_1, ..., k_n$ so that $|f(x) - f(k_i)| < r_{k_i}/2$. We then compute that

$$|g(x) - f(k_i)| \leq |g(x) - f(x)| + |f(x) - f(k_i)| < \epsilon + r_{k_i}/2 \leq r_{k_i}.$$

It follows that $g(x) \in U$ and $g(K) \subseteq U$ so $g \in T(K, U)$ as desired. We therefore have that $T(K, U)$ is open in the norm induced topology. This shows that the "identity" map from $C(X)$ with the norm topology to $C_{co}(X)$ is continuous.

We now claim that the "identity map" from $C_{co}(X)$ to $C(X)$ with the the norm topology is continuous. Suppose $f \in C(X)$ and there is a net $\alpha \to f$ in the compact-open topology. We must show that $\alpha \to f$ in norm. Fix $\epsilon > 0$. For each $x \in X$, find open $U_x \subseteq X$ so that $f(U_x) \subseteq B_{\epsilon/4}(f(x))$. Since X is compact, we may find $x_1, ..., x_n$ so that $\{U_{x_1}, ..., U_{x_n}\}$ covers X. Further, observe that for each $x \in X$,

$$f(\overline{U_x}) \subseteq \overline{f(U_x)} = \overline{B_{\epsilon/4}(f(x))} \subseteq B_{\epsilon/2}(f(x))$$

and thus $f \in T(\overline{U_x}, B_{\epsilon/2}(f(x)))$. Further, since X is compact and $\overline{U_x}$ closed, we have $\overline{U_x}$ is compact. Since $\alpha \to f$ in the compact-open topology, we have that $\alpha \in_{\text{ev}} \bigcap_{i=1}^n T(\overline{U_{x_i}}, B_{\epsilon/2}(f(x_i)))$. Suppose that $g \in \bigcap_{i=1}^n T(\overline{U_{x_i}}, B_{\epsilon/2}(f(x_i)))$ and $x \in X$. We have some $i = 1, ..., n$ so that $x \in U_{x_i}$. Thus, we have that $g(x), f(x) \subseteq B_{\epsilon/2}(f(x_i))$. Therefore, $|f(x) - g(x)| < \epsilon$. From this calculation, we have that $\alpha \to f$ in norm. We now have that the "identity map" from $C_{co}(X)$ to $C(X)$ with the the norm topology is continuous. Therefore, $C_{co}(X)$ and $C(X)$ with supremum norm are identical.

It remains to show that $C_{co}(X)$ is complete. Suppose $\alpha : \mathbb{N} \to C(X)$ is Cauchy. For each $x \in X$, we have that $\alpha(x)$ is Cauchy. By completeness of $\mathbb{K}$, each $\alpha(x)$ converges to some unique value $f(x) \in \mathbb{K}$, defining a function $f : X \to \mathbb{K}$. We will argue that $f \in C(X)$ and that $\alpha \to f$ in norm.

We prove first that $\alpha \to f$ in norm. Fix $\epsilon > 0$. We may find $N \in \mathbb{N}$ so that for all $n, m \geq N$ we have $||\alpha_n - \alpha_m|| < \epsilon/2$. It follows that for all $x \in X$ that

$$|\alpha_n(x) - f(x)| = \lim_{s \to \infty} |\alpha_n(x) - \alpha_{m+s}(x)| \leq \epsilon/2 < \epsilon.$$

We thus have that $\alpha \to f$ in norm. We may finally conclude that f is continuous since it is a uniform limit of continuous functions. QED

3.4 C-Embedded Spaces

In functional analysis, many important questions and properties concern the bidual of a vector space. Here, we call the function space $C(X)$ the paradual. This parallel suggests considering the paradual of a paradual, which is the subject of this section.

Definition 3.4.1. For any convergence space X and $x \in X$, define $\mathrm{ev}_x \in CC(X)$ by $\mathrm{ev}_x(f) = f(x)$ for all $f \in C(X)$. Define then $i_X : X \to C_cC_c(X)$ by $i_X(x) = \mathrm{ev}_x$ for all $x \in X$.

Remark 3.4.2. Note that the evaluation at a point map ev_x is merely a restriction of the evaluation map. This justifies the continuity of ev_x and thus well definition of i_X.

Lemma 3.4.3. *If X is any convergence space, then i_X is continuous.*

Proof. This follows from the observation that the transpose of i_X is the evaluation map $\mathrm{ev} : X \times C_c(X)$ which is continuous. QED

Definition 3.4.4. A convergence space X is called *c-embedded* when $i_X : X \to C_c(C_c(X, \mathbb{R}), \mathbb{R})$ is an embedding.

Note that here we specify the ground field as $\mathbb{R}$. This is necessary as it is not at all clear that the analogous definition using $\mathbb{C}$ would be equivalent. We will see that in fact the c-embeddedness of a space is independent of the ground field. We will first need a lemma.

Lemma 3.4.5. *Suppose X is functionally Hausdorff and $C \subseteq X$ is weakly closed. If $U \subseteq X$ is such that for some $\epsilon > 0$ we have $T(C, \epsilon\mathbb{D})(U) \subseteq \Delta$, then $U \subseteq C$.*

Proof. Suppose for the sake of contradiction that there is $x \in U$ with $x \notin C$. Since X is functionally Hausdorff, we have by Corollary 2.6.45 that X_σ is a Tychonoff space. We may thus find a continuous map $f : X \to [0,1]$ so that $f(C) = 0$ and $f(x) = 1$. We may extend the codomain of f to $\mathbb{K}$. We then have that $f(C) \subseteq \epsilon\mathbb{D}$, so that $f \in T(C, \epsilon\mathbb{D})$, but $f(x) = 1 \notin \mathbb{D}$. We thus have that $T(C, \epsilon\mathbb{D})(U) \not\subseteq \mathbb{D}$. We have obtained a contradiction and conclude the proof. QED

We may now describe c-embedded spaces in more familiar terms.

Theorem 3.4.6. *Suppose X is a convergence space. The map $i_X : X \to C_cC_c(X)$ is an embedding if and only if X is Choquet, functionally regular, and functionally Hausdorff.*

Proof. Suppose i_X is an embedding. We have that $C_cC_c(X)$ is Choquet, functionally regular, and functionally Hausdorff by Proposition 3.2.16 since $\mathbb{K}$ has these properties and thus by Proposition 2.4.18 that X has these properties since it is homeomorphic to a subspace of $C_cC_c(X)$.

For the other direction, assume now that X is Choquet, functionally regular, and functionally Hausdorff. We will first prove that i_X is an injection. Suppose $x, y \in X$ with $i_X(x) = i_X(y)$. This means that for all continuous functions $f : X \to \mathbb{K}$ we have $f(x) = f(y)$. We then have that for all $f \in C(X)$ that $f([x]) \to f(y)$ and thus that $[x] \to_\sigma y$. Equally well, we have $[x] \to_\sigma x$ and since X is functionally Hausdorff, we see that $x = y$.

Now, we will prove that i_X^{-1} is continuous. Suppose $\mathcal{F}$ is a filter on X so that $i_X(\mathcal{F}) \to i_X(x)$ for some $x \in X$. We aim to show that $\mathcal{F} \to x$. Since X is Choquet, it suffices to show that $\mathcal{U} \to x$ for all ultrafilters $\mathcal{U} \supseteq \mathcal{F}$. Suppose $\mathcal{U}$ is such an ultrafilter.

Suppose to contradiction that $\mathcal{U}$ fails to converge. For every converging filter $\mathcal{G}$ on X, we have that $\mathcal{U} \not\supseteq \overline{\mathcal{G}}^\sigma$. There is then $G_\mathcal{G} \in \mathcal{G}$ with $\overline{G_\mathcal{G}}^\sigma \notin \mathcal{U}$. We then have that

$$\mathcal{C} = \{\overline{G_\mathcal{G}}^\sigma : \mathcal{G} \text{ is a converging filter on } X\}$$

is a covering system for X by σ-closed sets none of which are contained in $\mathcal{U}$.

Define a filter on $C(X)$ by

$$\mathscr{F} = [\{T(C, \epsilon\mathbb{D}) : C \in \mathcal{C} \ \wedge \ \epsilon > 0\}].$$

Since every filter converging in X contains some $C \in \mathcal{C}$, we obtain that $\mathscr{F} \to 0$ in $C_c(X)$. It then follows that $i_X(\mathcal{U})(\mathscr{F}) \to 0$ in $\mathbb{K}$. There is then some $U \in \mathcal{U}$ and $F \in \mathscr{F}$ so that $i_X(U)(F) = F(U) \subseteq \mathbb{D}$. Indeed, we may assume without loss of generality

$$F = T(C_1, \epsilon_1\mathbb{D}) \cap T(C_2, \epsilon_2\mathbb{D}) \cap \cdots \cap T(C_n, \epsilon_n\mathbb{D}) \supseteq T(C_1 \cup C_2 \cup \cdots \cup C_n, \min_{1 \le i \le n} \epsilon_i\mathbb{D})$$

Further, since $C_1 \cup C_2 \cup \cdots \cup C_n$ is σ-closed, we have by Lemma 3.4.5 that $U \subseteq C_1 \cup C_2 \cup \cdots \cup C_n$. But then $C_1 \cup C_2 \cup \cdots \cup C_n \in \mathcal{U}$. But, since $\mathcal{U}$ is an ultrafilter and $C_1, ..., C_n \notin \mathcal{U}$, we have that $X \setminus C_1, ..., X \setminus C_n \in \mathcal{U}$. This leads to $\emptyset \in \mathcal{U}$, a contradiction. This contradiction shows that $\mathcal{U}$ converges to some $y \in X$. But since $i_X(\mathcal{U}) \supseteq i_X(\mathcal{F})$, we have that $i_X(y) = i_X(x)$ and $x = y$ by injectivity. Since X is Choquet, we have $\mathcal{F} \to x$. QED

Theorem 3.4.6 produces many useful corollaries.

We have that the conditions necessary and sufficient for X to embed into $CC(X)$ are independent of the value of the ground field $\mathbb{K}$. This immediately gives

Corollary 3.4.7. *A convergence space X is c-embedded if and only if $i_X : X \to CC(X)$ is an embedding regardless of the value of the ground field.*

C-embeddedness is characterized by functional Hausdorffness, functional regularity, and the Choquet property and by Proposition 2.4.18 and Proposition 2.6.5 (taking M to be the Choquet modification) these properties pass to subspaces. This gives

Corollary 3.4.8. *Subspaces of c-embedded spaces are c-embedded.*

Further, recalling from Corollary 3.2.14 and Proposition 3.2.16 that $CC(X)$ is functionally Hausdorff, functionally regular, and Choquet since $\mathbb{K}$ is, we obtain

Corollary 3.4.9. *For any convergence space X, the subspace $i_X(X)$ of $CC(X)$ is c-embedded.*

Further still, it is an easy check that functional regularity, functional Hausdorffness, and the Choquet property are all preserved under isomorphism. Thus,

Corollary 3.4.10. *Any space homeomorphic to a c-embedded space is c-embedded.*

Since all topological spaces are Choquet, we obtain

Corollary 3.4.11. *A topological space is c-embedded if and only if it is functionally regular and functionally Hausdorff.*

Up until this point, we have given no explicit examples of c-embedded spaces. Now, Theorem 3.4.6 allows us to show that many familiar types of spaces are c-embedded.

Corollary 3.4.12. *Tychonoff spaces are c-embedded.*

Proof. Suppose X is a Tychonoff space. We immediately obtain that points may be separated by continuous functions, and thus X is functionally Hausdorff

Suppose $x \in X$ and U is an open neighborhood of x. We have that $x \notin X \setminus U$ which is closed. Let $f : X \to [0, 1]$ be a continuous function with $f(x) = 1$ and $f(X \setminus U) = \{0\}$. Define weakly closed $C = f^{-1}([1/2, 1]) \supseteq X \setminus U$. We then have

$$\{x\} \subseteq X \setminus C \subseteq U$$

Further, suppose α is a convergent net in $X \setminus C$. We have that $f(\alpha) \geq 1/2$. Thus,

$$\{x\} \subseteq X \setminus C \subseteq \overline{X \setminus C}^{\sigma} \subseteq U.$$

This shows that $\overline{\mathcal{N}_x}^{\sigma} \supseteq \mathcal{N}_x$ which means that X is functionally regular.

We conclude that X is c-embedded. QED

Thus, the realm of c-embedded spaces include

- $\mathbb{R}$ and $\mathbb{C}$;
- All metric spaces;
- All compact Hausdorff spaces;
- All normal spaces.

Even though the only topological c-embedded spaces discussed thus far are Tychonoff, there are non Tychonoff c-embedded topological spaces. In [BM76], the authors classify all c-embedded topological spaces in more traditional topological terms.

The last corollary to Theorem 3.4.6 we will give describes compact c-embedded spaces.

Corollary 3.4.13. *All compact c-embedded spaces are topological.*

Proof. All c-embedded spaces are regular, Hausdorff, and Choquet. Since compact, Choquet, regular, Hausdorff spaces are topological by Corollary 2.6.39, the result follows. QED

3.5 C-Embedded Modification

As in Section 2.6 with Choquet, pretopological, topological, and Tychonoff spaces, there is a modification which produces a c-embedded space from any convergence space. The idea is to leverage Corollary 3.4.9.

Proposition 3.5.1. *For any convergence spaces X and Y and continuous $f : X \to Y$ define $c(X) = i_X(X)$ with its subspace convergence structure inherited from $C_cC_c(X)$ and $c(f)$ be the domain-codomain restriction of $f^{**} : C_cC_c(X) \to C_cC_c(Y)$. With this definition, (c, i) is a modification of convergence spaces where $i : id \to c$ is the natural transformation with component i_X at each convergence space X.*

Proof. We first check that c is a functor from **CONV** $\to$ **CONV**. By Corollary 3.2.21, we have that f^{**} is continuous, so we really need only check that if $f : X \to Y$ is continuous, then $f^{**}(i_X(X)) \subseteq i_Y(Y)$. This holds as for all $x \in X$ and $h \in C(Y)$

$$\begin{aligned} f^{**}(i_X(x))(h) &= f^{**}(\mathrm{ev}_x)(h) \\ &= \mathrm{ev}_x \circ f^*(h) \\ &= \mathrm{ev}_x(h \circ f) \\ &= h \circ f(x) \\ &= \mathrm{ev}_{f(x)}(h) \\ &= i_Y(f(x))(h) \end{aligned}$$

so that $f^{**}(i_X(x)) = i_Y(f(x))$ as desired for $f^{**}(i_X(X)) \subseteq i_Y(Y)$.

If $f : X \to Y$ is a continuous mapping of convergence spaces, then the commutativity of

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ c(X) & \xrightarrow[c(f)]{} & c(Y) \end{array}$$

follows from the previous computation. Further, we have that each component of i is continuous. Thus, i is a natural transformation.

We last need to check that i_X^{-1} is continuous if there is a convergence space Y with $X \cong c(Y)$. We have by Corollary 3.4.9 that $c(Y)$ is c-embedded, so by Corollary 3.4.10, X is c-embedded and so i_X (or rather its codomain restriction) is a homeomorphism. QED

The last result of this section will be that the paradual of a space is completely determined by the paradual of its c-embedded modification.

Proposition 3.5.2. *If X and Y are convergence spaces, the map $c_{X,Y} : C(X,Y) \to C(c(X), c(Y))$ given by $f \mapsto c(f)$ is continuous.*

Proof. This follows from the observation that

$$\begin{array}{ccc} C(X,Y) \times c(X) & \overset{T_1(c_{X,Y})}{\searrow} & \\ {\scriptstyle (\cdot)^{**}\times \mathrm{id}}\downarrow & & \\ C(c(X), c(Y)) \times c(X) & \xrightarrow[\mathrm{ev}]{} & c(Y) \end{array}$$

Commutes. Thus, $T_1(c_{X,Y})$ is continuous as a composition of continuous functions. QED

Theorem 3.5.3. *If X and Y are convergence spaces and Y is c-embedded, then*

$$C(X,Y) \cong C(c(X), Y).$$

Proof. Since Y is c-embedded, we have that $i_Y : Y \to c(Y)$ is a homeomorphism. We thus have that $i_{Y*} : C(c(X), Y) \to C(c(X), c(Y))$ is a homeomorphism by functoriality. We then check that

$$\begin{array}{ccc} C(c(X),Y) & \xrightarrow{\quad i_X^* \quad} & C(X,Y) \\ & {\scriptstyle i_{Y*}} \searrow \quad \swarrow {\scriptstyle c_{X,Y}} & \\ & C(c(X),c(Y)) & \end{array}$$

commutes. Let $f \in C(c(X), Y)$. We claim that

$$i_{Y*}(f) = c_{X,Y} \circ i_X^*(f)$$

To see this, fix any $x \in X$ and $h \in C(Y)$. We observe

$$\begin{aligned} i_{Y*}(f)(i_X(x))(h) &= (i_Y \circ f)(i_X(x))(h) \\ &= i_Y(f(i_X(x)))(h) \\ &= h(f(i_X(x))) \end{aligned}$$

and

$$\begin{aligned} (c_{X,Y} \circ i_X^*)(f)(i_X(x))(h) &= c_{X,Y}(f \circ i_X)(i_X(x))(h) \\ &= (f^{**} \circ i_X^{**})(i_X(x))(h) \\ &= (f^{**}(i_X(x) \circ i_X^*))(h) \\ &= (i_X(x) \circ i_X^* \circ f^*)(h) \\ &= i_X(x)(h \circ f \circ i_X) \\ &= h(f(i_X(x))) \end{aligned}$$

as desired for commutativity. This then shows that i_X^* has inverse $(i_{Y*})^{-1} \circ c_{X,Y}$. QED

Chapter 4

Convergence Vector Spaces

In this chapter, we investigate convergence vector spaces, the convergence analogue to topological vector spaces (Appendix D), closely following the exposition in [BB02]. Many of the properties of convergence vector spaces can be discussed purely in the context of the underlying group. We thus explore groups equipped with convergence structures in the first section. In the next, we consider convergence vector spaces and various properties and constructions. In the third section, we specialize to locally convex vector spaces and learn how to turn any convergence vector space into a locally convex topological vector space. In the last section, we discuss some basic properties of duality.

4.1 Convergence Groups

Definition 4.1.1. A *convergence group* is a group G so that the group operation $+ : G \times G \to G$ and inversion $-(\cdot) : G \to G$ are continuous.

Remark 4.1.2. Since we study convergence groups with the goal of learning about convergence vector spaces, we will write our groups additively. This will make the notational transition to vector spaces smoother. Despite this notational choice, we will not assume that these groups are abelian.

From Definition 4.1.1 we immediately obtain several other useful continuous maps.

Lemma 4.1.3. *If G is a group and $g \in G$,*

1. *the difference map $- : G \times G \to G$ given by $(h, k) \mapsto h - k$ is continuous;*

2. *the translation map $g + (\cdot) : G \to G$ given by $h \mapsto g + h$ is continuous;*

3. *the translation map $(\cdot) + g : G \to G$ given by $h \mapsto h + g$ is continuous.*

Remark 4.1.4. The maps $g + (\cdot) : G \to G$ and $(\cdot) + g : G \to G$ are homeomorphisms with inverses given by $-g + (\cdot) : G \to G$ and $(\cdot) - g : G \to G$ respectively.

Defining convergence groups as in Definition 4.1.1 will make the parallel to convergence vector spaces (Definition 4.2.1) perfectly clear. However, verifying that a particular group satisfies the definition may be expedited by using the following result

Proposition 4.1.5. *A group G is a convergence group if and only if the difference map $- : G \times G \to G$ given by $(h, k) \mapsto h - k$ is continuous.*

Proof. In light of Definition 4.1.1 and Lemma 4.1.3, we need only verify that the continuity of the difference implies continuity of inversion and addition. For this, it is enough to note that $-g = 0 - g$ and $g + h = g - (-h)$ for all $g, h \in G$ so that inversion and addition are merely compositions of continuous functions. QED

Several properties of convergence groups are determined by the behavior of the identity element.

Proposition 4.1.6. *A group homomorphism $\varphi : G \to H$ of convergence groups is continuous if and only if it is continuous at 0.*

Proof. Certainly, if φ is continuous, it is continuous at 0. Now, suppose φ is continuous at 0. Fix $g \in G$ and a filter $\mathcal{F} \to g$. We then have that $\mathcal{F} - g \to 0$ and $\varphi(\mathcal{F} - g) \to 0$ by continuity at 0. We see that $\varphi(\mathcal{F} - g) = \varphi(\mathcal{F}) - \varphi(g)$. Therefore, adding $\varphi(g)$ to both sides, we obtain $\varphi(\mathcal{F}) \to \varphi(g)$. QED

Proposition 4.1.7. *A convergence group G is Hausdorff if and only if $\{0\}$ is closed.*

Proof. Suppose G is Hausdorff. Any net in $\{0\}$ is constant and thus converged to 0. Since G is Hausdorff, this limit is unique. Thus, $a(\{0\}) = \{0\}$ and we have that $\{0\}$ is closed.

Now, suppose that $\{0\}$ is closed. Suppose α is a net in G so that $\alpha \to g, h$ for some $g, h \in G$. We have that $\alpha - \alpha \to g - h$. We have that $\alpha - \alpha$ has subnet β which is constantly 0. We then have that $\beta \to g - h$. We then have that $g - h \in a(\{0\}) = \{0\}$. Therefore, $g = h$ and we have that G is Hausdorff. QED

Proposition 4.1.8. *Any pretopological convergence group is topological.*

Proof. Suppose G is a pretopological convergence group with $S \subseteq G$ and $g \in a(a(S))$. Suppose N is a vicinity of g. We have that $\mathcal{V}_g \to g$ and $\mathcal{V}_0 \to 0$ so that $\mathcal{V}_g + \mathcal{V}_0 \to g$ and $\mathcal{V}_g + \mathcal{V}_0 \supseteq \mathcal{V}_g$. Therefore, there are vicinities U of g and V of 0 so that $N \supseteq U + V$.

Since $g \in a(a(S))$, there is a filter $\mathcal{F} \to g$ with $a(S) \in \mathcal{F}$. We thus have that $\mathcal{F} \supseteq \mathcal{V}_g$. Therefore, $U \in \mathcal{F}$ and $a(S) \cap U \neq \emptyset$. There is, therefore, some $h \in a(S) \cap U$ witnessed by filter $\mathcal{H} \to h$ with $S \in \mathcal{H} \supseteq \mathcal{V}_h$. We have by Corollary 2.2.26 that $h + V \in \mathcal{V}_h$. Therefore, $S \cap (h + V) \neq \emptyset$.

Since $h \in U$, we now have that $S \cap (U + V) \neq \emptyset$. Therefore,

$$N \cap S \supseteq S \cap (U + V) \neq \emptyset.$$

As this holds for all vicinities N of g, we may consider the filter $\mathcal{V}_g \cap S \supseteq \mathcal{V}_g$. Thus, since G is pretopological, we have that $\mathcal{V}_g \cap S \to g$ Since $S \in \mathcal{V}_g \cap S$, we have that $g \in a(S)$. We conclude that $a(a(S)) = a(S)$. As G is pretopological with idempotent adherence, we have by Theorem 2.6.11 that G is topological. QED

Convergence groups interact well with initial and quotient convergence structures.

Proposition 4.1.9. *Let G be a group and $\{G_i\}_{i\in I}$ be a family of convergence groups so that for each $i \in I$ there is a group homomorphism $\varphi_i : G \to G_i$. Then the initial convergence structure on G with respect to the φ_i is a group convergence structure.*

Proof. For each $i \in I$, consider the following diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\varphi_i \times \varphi_i} & G_i \times G_i \\ {\scriptstyle -}\big\downarrow & & \big\downarrow{\scriptstyle -} \\ G & \xrightarrow[\varphi_i]{} & G_i \end{array}$$

which commutes because φ_i is a homomorphism. The initial convergence structure assures us that the φ_i are continuous. Thus, the "top path" is continuous as a composition. We then obtain that $\varphi_i \circ -$ is continuous for all $i \in I$ which shows that $- : G \times G \to G$ is continuous via Proposition 2.3.5 and that G is a convergence group by Proposition 4.1.5. QED

Lemma 4.1.10. *If G is a convergence group and N a normal subgroup, equip G/N with the final convergence structure over the quotient map $q : G \to G/N$,* i.e. *the quotient convergence structure over q. A filter $\mathcal{F}$ on G/N converges to $g + N$ if and only if there is a filter $\mathcal{G}$ on G converging to g so that $\mathcal{F} \supseteq q(\mathcal{G})$.*

Proof. The sufficiency of this condition is evident in light of Definition 2.3.23. We prove necessity. Suppose $g \in G$ and there is a filter $\mathcal{F} \to g + N$ in G/N. Then by Proposition 2.3.28 we have that there are $g_1, ..., g_n \in g + N$ so that for each $i = 1, ..., n$ there are filters $\mathcal{F}_i \to g_i$ in G so that $\mathcal{F} \supseteq \bigcap_{i\in I} q(\mathcal{F}_i)$. We then have that $\mathcal{F}_i - g_i + g \to g$ by Lemma 4.1.3. Let $\mathcal{G} = \bigcap_{i=1}^n \mathcal{F}_i - g_i + g$. We have that $\mathcal{G} \to g$. Further, we have that

$$q(\mathcal{G}) = \bigcap_{i=1}^{n} q(\mathcal{F}_i - g_i + g) = \bigcap_{i=1}^{n} q(\mathcal{F}_i).$$

Therefore, $\mathcal{F} \supseteq q(\mathcal{G})$ as desired. QED

Remark 4.1.11. This argument works just as well when the quotient is realized by some surjective group homomorphism rather than quotienting out a subgroup.

Proposition 4.1.12. *If G is a convergence group and N a normal subgroup of G, then G/N with the quotient convergence structure over the quotient map $q : G \to G/N$ is a convergence group.*

Proof. By Proposition 4.1.5, we need only check that $- : G/N \times G/N \to G/N$ is continuous. Let $g, h \in G$ and $\mathcal{H}$ be a filter on $G/N \times G/N$ with $\mathcal{H} \to (g+N, h+N)$. There are then filters $\mathcal{H}_1, \mathcal{H}_2$ on G/N so that $\mathcal{H}_1 \to g + N$ and $\mathcal{H}_2 \to h + N$ and $\mathcal{H} \supseteq \mathcal{H}_1 \times \mathcal{H}_2$. By Lemma 4.1.10, we have filters $\mathcal{F} \to g$ and $\mathcal{G} \to h$ in G and $\mathcal{H}_1 \supseteq q(\mathcal{F})$ and $\mathcal{H}_2 \supseteq q(\mathcal{G})$. We know that $\mathcal{F} - \mathcal{G} \to g - h$. By continuity of the quotient, we have

$$q(\mathcal{F}) - q(\mathcal{G}) = q(\mathcal{F} - \mathcal{G}) \to g - h + N.$$

Since

$$\mathcal{H} \supseteq \mathcal{H}_1 \times \mathcal{H}_2 \supseteq q(\mathcal{F}) \times q(\mathcal{G}),$$

we now have that $-\mathcal{H} \to g - h + N$ so that $- : G/N \times G/N \to G/N$ is continuous. We conclude that G/N is a convergence group. QED

The topological properties of a normal subgroup determine many properties of the quotient space.

Proposition 4.1.13. *Suppose G is a convergence group with normal subgroup N.*

(a) The quotient G/N is Hausdorff if and only if N is closed.

(b) The quotient G/N is discrete[4] *if and only if N is open.*

Proof. We show (a) first. By Proposition 4.1.7, it suffices to show that N being closed in G is equivalent to $\{N\}$ closed in G/N.

Certainly if $\{N\}$ is closed, then its preimage N under the continuous quotient map $q : G \to G/N$ is also closed. Conversely, assume that $\{N\}$ is not closed. Thus, there is some $g \in G$ so that $[N] \to g + N$ but $g \notin N$. We find a filter $\mathcal{F} \to g$ in G so that $[N] \supseteq q(\mathcal{F})$. It follows that for each $F \in \mathcal{F}$ we have $F \cap N \neq \emptyset$. We have that $\mathcal{F} \cap N \supseteq \mathcal{F}$ so that $\mathcal{F} \cap N \to g$ and $N \in \mathcal{F} \cap N$. Therefore, $g \in a(N)$. As it has been established that $g \notin N$, we have that $a(N) \neq N$. So N is not closed.

We now prove (b). Suppose G/N is discrete, that is only the point filters converge. Suppose $g \in N$ and there is a filter $\mathcal{F} \to g$. We have that $q(\mathcal{F}) \to N$. Therefore, $q(\mathcal{F}) = [N]$. There is then $F \in \mathcal{F}$ with $q(F) = N$. This is only possible if $D \subseteq N$. Therefore, $N \in \mathcal{F}$. This holds for all $g \in N$ and $\mathcal{F} \to g$, so we have that N is open.

Now, suppose that N is open. Suppose $\mathcal{F} \to gN$ in G/N. Then $\mathcal{F} - g + N \to N$. There is then some filter $\mathcal{G}$ converging to 0 so that $\mathcal{F} - g + N \supseteq q(\mathcal{G})$. Since $N \ni 0$ is open, we have that $N \in \mathcal{G}$ and thus $q(\mathcal{G}) = [N]$. Therefore, $\mathcal{F} - g + N = [N]$ and $\mathcal{F} = [g + N]$. Therefore, all converging filters are point filters and $\mathcal{G}/N$ is discrete. QED

[4]The only converging filters are point filters, and these only converge to their generating point.

We recall from Theorem 2.3.31 that quotients of topological convergence spaces need not be topological. However, for groups, quotients are more well behaved.

Theorem 4.1.14. *If G is topological and N a normal subgroup of G, then G/N is topological.*

Proof. By Proposition 4.1.8, it suffices to check that G/N is pretopological. Fix $g \in G$ and V a vicinity of g. Suppose $\mathcal{F} \to g + N$ in G/N. By Lemma 4.1.10 we have some filter $\mathcal{G} \to g$ so that $\mathcal{F} \supseteq q(\mathcal{G})$. We have then that $V \in \mathcal{G}$ since V is a vicinity of g. Therefore, $q(V) \in \mathcal{F}$. This shows, $q(V)$ is a vicinity of $g + N$. We then have that $\mathcal{V}_{g+N} \supseteq q(\mathcal{V}_g)$. Since G is topological, $\mathcal{V}_g \to g$, and by continuity of q, we have $q(\mathcal{V}_g) \to g+N$. Therefore, $\mathcal{V}_{g+N} \to g+N$, so that the vicinity filters in G/N converge and G/N is pretopological. QED

4.2 Convergence Vector Spaces

Definition 4.2.1. A vector space V is called a *convergence vector space* or CVS when it is equipped with a convergence structure so that vector addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \mathbb{K} \times V \to V$ are continuous.

Remark 4.2.2. By comparing the above definition to Definition 4.1.1, it is clear that all convergence vector spaces are convergence groups. Further, all topological vector spaces, see Definition D.2.1, are convergence vector spaces.

We will now present a key class of convergence vector spaces, function spaces with codomain a convergence vector space.

Lemma 4.2.3. *If X is a convergence space, V a CVS, $\lambda \in \mathbb{K}$, and $f, g \in C(X, V)$, then*

1. *$f + g \in C(X, V)$;*
2. *$\lambda f \in C(X, V)$.*

Proof. This follows from the fact that the following diagrams commute

$$\begin{array}{ccccc} X & \xrightarrow{\Delta} & X \times X & \xrightarrow{f \times g} & V \times V \\ & & & \searrow^{f+g} & \downarrow + \\ & & & & V \end{array} \qquad \begin{array}{ccccc} X & \xrightarrow{\Delta} & X \times X & \xrightarrow{\lambda^* \times f} & \mathbb{K} \times V \\ & & & \searrow^{\lambda f} & \downarrow \cdot \\ & & & & V \end{array}$$

where $\Delta : X \to X \times X$ is given by $x \mapsto (x, x)$ which is continuous by the universal property of products and $\lambda^* : X \to \mathbb{K}$ is given by $x \mapsto \lambda$ which is continuous since any constant function is continuous. QED

Proposition 4.2.4. *If X is a convergence space and V a convergence vector space, then $C(X, V)$ is also a convergence vector space when endowed with the usual pointwise operations.*

Proof. We must show that $+ : C(X,V) \times C(X,V) \to C(X,V)$ and $\cdot : \mathbb{K} \times C(X,V) \to C(X,V)$ are continuous. We consider their transposes. We see that $T_1(+)$ (and thus $+$) is continuous by noting that

$$\begin{array}{ccccc}
C(X,V) \times C(X,V) \times X & & \xrightarrow{\quad T_1(+) \quad} & & V \\
\Big\downarrow{\scriptstyle \mathrm{id} \times \mathrm{id} \times \Delta} & & & & \Big\uparrow{\scriptstyle +} \\
C(X,V) \times C(X,V) \times X \times X & \xrightarrow[\cong]{} & C(X,V) \times X \times C(X,V) \times X & \xrightarrow[\mathrm{ev} \times \mathrm{ev}]{} & V \times V
\end{array}$$

commutes. Likewise for scalar multiplication, we see that

$$\begin{array}{ccc}
\mathbb{K} \times C(X,V) \times X & & \\
\Big\downarrow{\scriptstyle \mathrm{id}_{\mathbb{K}} \times \mathrm{ev}} & \searrow^{T_1(\cdot)} & \\
\mathbb{K} \times V & \xrightarrow[\cdot]{} & V
\end{array}$$

commutes as needed to show scalar multiplication is continuous. QED

Corollary 4.2.5. *If X is a compact convergence space, then $C(X)$ is a Banach space under supremum norm.*

Proof. Since X is compact and $i_X : X \to c(X)$ is a continuous surjection, we have that $c(X)$ is compact. Since $c(X)$ is also c-embedded, it is topological by Corollary 3.4.13. We then have $C(X) \cong C(c(X)) = \mathfrak{C}C_{co}(c(X))$ by Theorem 3.5.3 and Theorem 3.3.6. Further, by this same result, $\mathfrak{C}C_{co}(c(X))$ is a Banach space under supremum norm.

It is not difficult to check that the isomorphism of Theorem 3.5.3 is linear, from which it follows that $C(X)$ is a Banach space with norm

$$||f|| := \sup_{z \in c(X)} |i_{\mathbb{K}^*}^{-1} \circ c_{X,\mathbb{K}}(f)(z)|$$

To show that $||f|| = \sup_{x \in X} |f(x)|$, it suffices to show that $i_X^* : C(c(X)) \to C(X)$ is norm preserving. Let $f \in C(c(x))$ and $x \in X$. We have that $i_X^*(f)(x) = f \circ i_X(x) = f(\mathrm{ev}_x)$. Thus,

$$\sup_{x \in X} |i_X^*(f)(x)| = \sup_{x \in X} |f(\mathrm{ev}_x)| = \sup_{z \in c(X)} |f(z)|$$

as desired. QED

Notation 4.2.6. If V and W are convergence vector spaces, then we adopt the following notations

1. $\mathcal{L}(V,W)$ denotes the space of continuous linear maps from V to W viewed as a subspace of $C(V,W)$.

2. $\mathcal{L}(V) = \mathcal{L}(V,\mathbb{K})$.

3. $\mathbb{D}$ denotes the open unit ball in $\mathbb{K}$.

4. $\mathcal{D}$ denotes the neighborhood filter of 0 in $\mathbb{K}$. We note that $\mathcal{D} = [\{\epsilon\mathbb{D} : \epsilon > 0\}]$.

Since proofs involving convergence in a vector space often rely on both filters or nets and the linear structure, it is useful to know how these interact. The following lemma gathers together some useful equalities and containments. We phrase these in terms of filters and analogous formulations for nets may be obtained by passing through eventuality filters.

Lemma 4.2.7 (Filter Algebra)**.** *Let V, W be vector spaces over $\mathbb{K}$. Let $v, w \in V$ and $\alpha, \beta \in \mathbb{K}$ and $\mathcal{F}, \mathcal{G} \in \Phi(V)$ and $\mathcal{A}, \mathcal{B} \in \Phi(\mathbb{K})$ and $f : V \to W$ be a linear map. The following hold:*

1. $(\mathcal{F} + \mathcal{G}) + (v + w) = (\mathcal{F} + v) + (\mathcal{G} + w)$
2. $\alpha(\mathcal{F} + \mathcal{G}) = \alpha\mathcal{F} + \alpha\mathcal{G}$
3. $\alpha(\mathcal{F} + v) = \alpha\mathcal{F} + \alpha v$
4. $\mathcal{A}(\mathcal{F} + \mathcal{G}) \supseteq \mathcal{A}\mathcal{F} + \mathcal{A}\mathcal{G}$
5. $\mathcal{A}(\mathcal{F} + v) \supseteq \mathcal{A}\mathcal{F} + \mathcal{A}v$
6. $\mathcal{A}(\mathcal{B}v) = (\mathcal{A}\mathcal{B})v$
7. $\mathcal{D}\mathcal{D} = \mathcal{D}$
8. $\alpha\mathcal{D} = \mathcal{D}$ *if* $\alpha \neq 0$
9. $f(\mathcal{F} + \mathcal{G}) = f(\mathcal{F}) + f(\mathcal{G})$
10. $\mathcal{A}f(\mathcal{F}) = f(\mathcal{A}\mathcal{F})$
11. $\alpha f(\mathcal{F}) = f(\alpha\mathcal{F})$
12. $\mathcal{F} = \mathcal{F} + [0]$
13. $\mathcal{A}(v + w) \supseteq \mathcal{A}v + \mathcal{A}w$

Proof. We will consider each result in turn. For notational convenience, define $A : V \times V \to V$ to be vector addition and $m : \mathbb{K} \times V \to V$ to be scalar multiplication. For each $x \in V$, define $A_x : V \to V$ given by $y \mapsto y + x$. For each $\lambda \in \mathbb{K}$, define $m_\lambda : V \to V$ given by $y \mapsto \lambda y$.

1. Observe that $A_{v+w} \circ A = A \circ (A_v \times A_w)$. We then compute

$$\begin{aligned}
(\mathcal{F} + \mathcal{G}) + (v + w) &= A_{v+w} \circ A(\mathcal{F} \times \mathcal{G}) \\
&= A \circ (A_v \times A_w)(\mathcal{F} \times \mathcal{G}) \\
&= A(A_v(\mathcal{F}) \times A_w(\mathcal{G})) && \text{(Proposition 1.6.15)} \\
&= A((\mathcal{F} + v) \times (\mathcal{G} + v)) \\
&= (\mathcal{F} + v) + (\mathcal{G} + v).
\end{aligned}$$

This is the desired result for (1).

2. Observe that $m_\alpha \circ A = A \circ (m_\alpha \times m_\alpha)$. We then compute

$$\begin{aligned}\alpha(\mathcal{F}+\mathcal{G}) &= m_\alpha \circ A(\mathcal{F}\times\mathcal{G}) \\ &= A \circ (m_\alpha \times m_\alpha)(\mathcal{F}\times\mathcal{G}) \\ &= A(\alpha\mathcal{F}\times\alpha\mathcal{G}) && \text{(Proposition 1.6.15)} \\ &= \alpha\mathcal{F}+\alpha\mathcal{G}\end{aligned}$$

This is the desired result for (2).

3. Observe that $m_\alpha \circ A_v = A_{\alpha v} \circ m_\alpha$. We then compute

$$\begin{aligned}\alpha(\mathcal{F}+v) &= m_\alpha \circ A_v(\mathcal{F}) \\ &= A_{\alpha v} \circ m_\alpha(\mathcal{F}) \\ &= A_{\alpha v}(\alpha\mathcal{F}) \\ &= \alpha\mathcal{F}+\alpha v.\end{aligned}$$

This is the desired result for (3).

4. Suppose $U \in \mathcal{A}\mathcal{F}+\mathcal{A}\mathcal{G}$. We then have $A_1, A_2 \in \mathcal{A}$ and $F \in \mathcal{F}$ and $G \in \mathcal{G}$ so that

$$\begin{aligned}U &\supseteq A_1F + A_2G \\ &\supseteq (A_1\cap A_2)F + (A_1\cap A_2)G \\ &\supseteq (A_1\cap A_2)(F+G) \\ &\in \mathcal{A}(\mathcal{F}+\mathcal{G})\end{aligned}$$

We then have that $U \in \mathcal{A}(\mathcal{F}+\mathcal{G})$ as desired.

5. Suppose $U \in \mathcal{A}\mathcal{F}+\mathcal{A}v$. We then have $A_1, A_2 \in \mathcal{A}$ and $F \in \mathcal{F}$ so that

$$\begin{aligned}U &\supseteq A_1F + A_2v \\ &\supseteq (A_1\cap A_2)F + (A_1\cap A_2)v \\ &\supseteq (A_1\cap A_2)(F+v) \\ &\in \mathcal{A}(\mathcal{F}+v)\end{aligned}$$

We then have that $U \in \mathcal{A}(\mathcal{F}+v)$ as desired.

6. Let $U \in \mathcal{A}(\mathcal{B}v)$. We then have $A \in A$ and $B \in \mathcal{B}$ so that $U \supseteq A(Bv) = (AB)v$. Thus, $U \in (\mathcal{A}\mathcal{B})v$. The other inclusion follows in like manner.

7. Suppose $U \in \mathcal{D}^2$. We have that $U \supseteq D_1D_2$ for some $D_1, D_2 \in \mathcal{D}$. We then have some $\epsilon_1, \epsilon_2 > 0$ so that $D_1 \supseteq \epsilon_1\mathbb{D}$ and $D_2 \supseteq \epsilon_2\mathbb{D}$ and $U \supseteq (\epsilon_1\mathbb{D})(\epsilon_2\mathbb{D})$. If $z_1, z_2 \in \mathbb{K}$ with $|z_1| < \epsilon_1$ and $|z_2| < \epsilon_2$, then $|z_1z_2| < \epsilon_1\epsilon_2$. Thus,

$$(\epsilon_1\mathbb{D})(\epsilon_2\mathbb{D}) \subseteq \epsilon_1\epsilon_2\mathbb{D}.$$

Further, if $z \in \epsilon_1\epsilon_2\mathbb{D}$, then $z = \epsilon_1\epsilon_2 s$ for some $s \in \mathbb{D}$. If $s = 0$, then $z = 0$ and $z \in (\epsilon_1\mathbb{D})(\epsilon_2\mathbb{D})$. Else, we then compute

$$\begin{aligned} z &= \epsilon_1\epsilon_2 s \\ &= \epsilon_1\epsilon_2|s|\frac{s}{|s|} \\ &= \left(\epsilon_1|s|^{1/2}\right)\left(\epsilon_2|s|^{1/2}\frac{s}{|s|}\right). \end{aligned}$$

Since $s \in \mathbb{D}$, we have $|s|^{1/2} < 1$. Thus, $\epsilon_1|s|^{1/2} \in \epsilon_1\mathbb{D}$ and similarly $\epsilon_2|s|^{1/2}\frac{s}{|s|} \in \epsilon_2\mathbb{D}$. Thus, $z \in (\epsilon_1\mathbb{D})(\epsilon_2\mathbb{D})$. Therefore,

$$(\epsilon_1\mathbb{D})(\epsilon_2\mathbb{D}) = \epsilon_1\epsilon_2\mathbb{D}.$$

So, $U \supseteq \epsilon_1\epsilon_2\Delta \in \mathcal{D}$ so that $U \in \mathcal{D}$. We conclude that $\mathcal{D}^2 \subseteq \mathcal{D}$.

Lastly, we have that $\mathcal{D}^2 \supseteq \mathcal{D}$ since multiplication is continuous in $\mathbb{K}$ and $\mathcal{D} \to 0$.

8. Since $\mathbb{K}$ is a topological vector space, we have that $\mathcal{D} \to 0$ and scalar multiplication is continuous. So, $\alpha\mathcal{D} \to 0$ which implies $\alpha\mathcal{D} \supseteq \mathcal{D}$. We also have that $\frac{1}{\alpha}\mathcal{D} \to 0$, so $\frac{1}{\alpha}\mathcal{D} \supseteq \mathcal{D}$. But then

$$\mathcal{D} = \alpha\frac{1}{\alpha}\mathcal{D} \supseteq \alpha\mathcal{D} \supseteq \mathcal{D}.$$

We then conclude that $\alpha\mathcal{D} = \mathcal{D}$ as desired for (8)

9. Observe that $f \circ A = A \circ (f \times f)$ since f is linear. Therefore,

$$\begin{aligned} f(\mathcal{F} + \mathcal{G}) &= f \circ A(\mathcal{F} \times \mathcal{G}) \\ &= A \circ (f \times f)(\mathcal{F} \times \mathcal{G}) \\ &= A(f(\mathcal{F}) \times f(\mathcal{G})) && \text{(Proposition 1.6.15)} \\ &= f(\mathcal{F}) + f(\mathcal{G}) \end{aligned}$$

as desired for (9).

10. We observe that $m \circ (\mathrm{id}_{\mathbb{K}} \times f) = f \circ m$ by linearity of f. Therefore,

$$\begin{aligned} \mathcal{A}f(\mathcal{F}) &= m \circ (\mathrm{id}_{\mathbb{K}} \times f)(\mathcal{A} \times \mathcal{F}) \\ &= f \circ m(\mathcal{A} \times \mathcal{F}) \\ &= f(\mathcal{A}\mathcal{F}) \end{aligned}$$

as desired for (10).

11. We observe that $m_\alpha \circ f = f \circ m_\alpha$ by linearity of f. Therefore,

$$\begin{aligned} \alpha f(\mathcal{F}) &= m_\alpha \circ f(\mathcal{F}) \\ &= f \circ m_\alpha(\mathcal{F}) \\ &= f(\alpha\mathcal{F}) \end{aligned}$$

as desired for (11).

12. Suppose $H \in \mathcal{F}+[0]$. There is then some $F \in \mathcal{F}$ and $\mathcal{G} \in [0]$ so that $H \supseteq F+G$. Since $0 \in G$, we have $F + G \supseteq F$. Thus, $H \supseteq F$ and $H \in \mathcal{F}$. This shows $\mathcal{F} \supseteq \mathcal{F} + [0]$.

    One the other hand, if $F \in \mathcal{F}$, then

$$F = F + \{0\} \in \mathcal{F} + [0]$$

    so that $\mathcal{F} \subseteq \mathcal{F} + [0]$.

13. Suppose $H \in \mathcal{A}v + \mathcal{A}w$. There is then some $S_1, S_2 \in \mathcal{A}$ so that $H \supseteq S_1 v + w S_2$. Then

$$H \supseteq S_1 v + S_2 w \supseteq (S_1 \cap S_2)(v + w) \in \mathcal{A}(v + w).$$

    Thus $\mathcal{A}(v + w) \supseteq \mathcal{A}v + \mathcal{A}w$.

QED

The convergence structure of a CVS is entirely dependent on the convergence of filters to 0. That is, a filter $\mathcal{F}$ on a CVS V converges to $v \in V$ if and only if $\mathcal{F} - v \to 0$. The following theorem enables us to define a convergence structure on a vector space V making V a CVS by merely specifying which filters converge to 0.

Theorem 4.2.8. *Let V be a vector space. Let Ψ be a collection of filters on V so that*

1. *If $\mathcal{F}, \mathcal{G} \in \Psi$ then $\mathcal{F} \cap \mathcal{G} \in \Psi$;*
2. *If $\mathcal{F} \in \Psi$ and $\mathcal{G} \supseteq \mathcal{F}$, then $\mathcal{G} \in \Psi$;*
3. *If $\mathcal{F}, \mathcal{G} \in \Psi$, then $\mathcal{F} + \mathcal{G} \in \Psi$;*
4. *If $\mathcal{F} \in \Psi$ then $\mathcal{D}\mathcal{F} \in \Psi$;*
5. *If $\mathcal{F} \in \Psi$ and $\alpha \in \mathbb{K}$ then $\alpha\mathcal{F} \in \Psi$.*
6. *If $v \in V$, then $\mathcal{D}v \in \Psi$.*

The relation $\mathcal{F} \to v$ if and only if $\mathcal{F} - v \in \Psi$ is a convergence structure on V making V a convergence vector space.

Proof. We must first check the three properties of convergence spaces.

1. First, note that $\mathcal{D}0 = [0] \in \Psi$. Thus, if $v \in V$, then $[v] - v = [0]$ and we have $[v] \to v$.
2. Suppose $\mathcal{F} \to v$ and $\mathcal{G} \supseteq \mathcal{F}$. We have that $\mathcal{G} - v \supseteq \mathcal{F} - v \in \Psi$ so that $\mathcal{G} \to v$.
3. Lastly, suppose that $\mathcal{F}, \mathcal{G} \to v$. We then have that $\mathcal{F} - v, \mathcal{G} - v \in \Psi$. It follows that $(\mathcal{F}-v)\cap(\mathcal{G}-v) \in \Psi$. However, we have that $(\mathcal{F}-v)\cap(\mathcal{G}-v) = \mathcal{F}\cap\mathcal{G}-v$. We conclude that $\mathcal{F} \cap \mathcal{G} \to v$.

We have established that this is a convergence structure. We now move to prove continuity of addition and scalar multiplication. Suppose we have filters $\mathcal{F}, \mathcal{G}$ on V so that $\mathcal{F} \times \mathcal{G} \to (v, w) \in V \times V$. We then have $\mathcal{F} - v, \mathcal{G} - w \in \Psi$. Then we have that

$$(\mathcal{F} + \mathcal{G}) - (v + w) = (\mathcal{F} - v) + (\mathcal{G} - w) \in \Psi,$$

so that $\mathcal{F} + \mathcal{G} \to v + w$, proving that addition is continuous.

Likewise for scalar multiplication, suppose that we have filters $\mathcal{N}, \mathcal{F}$ on $\mathbb{K}$ and V respectively so that $\mathcal{N} \times \mathcal{F} \to (\lambda, v) \in \mathbb{K} \times V$. We then have that $\mathcal{N} \to \lambda$ and $\mathcal{F} \to v$. This of course means that $\mathcal{F} - v \in \Psi$. Let $\mathcal{D}(\lambda)$ denote the neighborhood filter of λ. Since $\mathbb{K}$ is topological, we have that $\mathcal{N} \supseteq \mathcal{D}(\lambda)$. Further, observe that $\mathcal{D}(\lambda) = \mathcal{D} + \lambda$. Thus,

$$\begin{aligned}
\mathcal{N}\mathcal{F} - \lambda v &\supseteq (\mathcal{D} + \lambda)\mathcal{F} - \lambda v \\
&= (\mathcal{D} + \lambda)((\mathcal{F} - v) + v) - \lambda v \\
&\supseteq \mathcal{D}(\mathcal{F} - v) + \lambda(\mathcal{F} - v) + \mathcal{D}v + \lambda v - \lambda v \\
&= \mathcal{D}(\mathcal{F} - v) + \lambda(\mathcal{F} - v) + \mathcal{D}v \\
&\in \Psi.
\end{aligned}$$

We thus have that $\mathcal{N}\mathcal{F} - \lambda v \in \Psi$ as desired for $\mathcal{N}\mathcal{F} \to \lambda v$. We have thus shown that scalar multiplication is continuous. QED

Corollary 4.2.9. *If V is a vector space with convergence structure $\to$ and*

$$\Psi := \{\mathcal{F} \in \Phi(X) : \mathcal{F} \to 0\}$$

satisfies (1) - (6) of Theorem 4.2.8 and for each $v \in V$ the translation $(\cdot) + v : V \to V$ is continuous, then V with this convergence structure is a convergence vector space.

Proof. Define a convergence structure $\to_\Psi$ on V by $\mathcal{F} \to_\Psi v$ if and only if $\mathcal{F} - v \in \Psi$. By Theorem 4.2.8, this is a convergence vector space structure. For any $\mathcal{F} \in \Phi(V)$ and $v \in V$, one has

$$\begin{aligned}
\mathcal{F} \to_\Psi v &\iff \mathcal{F} - v \in \Psi \\
&\iff \mathcal{F} - v \to 0 \\
&\iff \mathcal{F} \to v
\end{aligned}$$

by continuity of $(\cdot) - v$ and $(\cdot) + v$. Thus, $\to$ and $\to_\Psi$ are identical and $(V, \to)$ is a CVS. QED

We now turn our attention to the initial and final convergence structures. We will see that the initial convergence structure over a family of linear maps into a family of convergence vector spaces will define a CVS. The final convergence structure given in Definition 2.3.23, however, is less well behaved and in general cannot be expected to produce a CVS. Thus, a new final vector space convergence structure will be given which will satisfy a universal property akin to Proposition 2.3.24. We will then see that quotient spaces defined by this new final structure will be exactly the same a quotients in the sense of Definition 2.3.27.

Proposition 4.2.10. *If $\{\varphi_i : V \to V_i\}$ is a family of linear maps from vector space V to convergence vector spaces V_i, then V is a convergence vector space when endowed with its initial convergence structure.*

Proof. That addition and scalar multiplication are continuous follows from Proposition 2.3.5 and the commutativity of the following diagrams for all $i \in I$

$$\begin{array}{ccc} V\times V & \xrightarrow{\varphi_i\times\varphi_i} & V_i\times V_i \\ {\scriptstyle +}\big\downarrow & & \big\downarrow{\scriptstyle +} \\ V & \xrightarrow[\varphi_i]{} & V_i \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{K}\times V & \xrightarrow{\mathrm{id}_{\mathbb{K}}\times\varphi_i} & \mathbb{K}\times V_i \\ {\scriptstyle \cdot}\big\downarrow & & \big\downarrow{\scriptstyle \cdot} \\ V & \xrightarrow[\varphi_i]{} & V_i \end{array}$$

QED

This means that we may obtain subspaces and products of convergence vector spaces in the expected way, simply by taking a subspace or product of the underlying convergence spaces and vector spaces individually.

Definition 4.2.11. Let V be a vector space and $\{\varphi_i : V_i \to V\}_{i\in I}$ a family of linear maps from convergence vector spaces V_i to V. Let Ω be the set of all vector space convergence structures on V making each φ_i continuous.[5] Denote by V_ω the vector space V equipped with convergence structure $\omega \in \Omega$. We define the *final vector space convergence structure on V relative to the $\{\varphi_i\}_{i\in I}$* to be the initial convergence structure relative to the family of inclusions $\{\iota_\omega : V \to V_\omega\}_{\omega\in\Omega}$.

Remark 4.2.12. The condition required for convergence in this final structure can be stated more plainly as

> A filter converges in V if and only if it converges in every convergence space V_ω so that each $\varphi_i : V_i \to V_\omega$ is continuous.

Stated this way, one can see that the final vector space convergence structure is the convergence structure on V making as few filters converge as possible but still making each $\varphi_i : V_i \to V$ continuous. The upshot to involving the initial convergence structure in the definition is that Proposition 4.2.10 assures us that V is a CVS when equipped with the final vector space convergence structure.

[5]Note that Ω is non-empty since it contains the *chaotic convergence structure*: all filters converge to all points.

As advertised, we have an analogue to Proposition 2.3.24

Proposition 4.2.13. *Let V be a vector space equipped with the final vector space convergence structure relative to $\{\varphi_i : V_i \to V\}_{i\in I}$, a family of linear maps from convergence vector spaces V_i to V. Then if W is a convergence vector space and $f : V \to W$ linear, then f is continuous if and only if the composition $f \circ \varphi_i$ is continuous for all $i \in I$.*

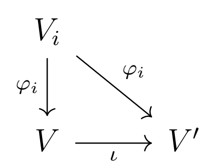
Proof. Certainly, if f is continuous, then $f \circ \varphi_i$ is continuous. Suppose then that $f \circ \varphi_i$ is continuous for each $i \in I$. Let V' denote V with the initial convergence structure relative to f and let $\iota : V \to V'$ be given by $x \mapsto x$. By Proposition 4.2.10, we have that V' is a CVS. Fix $i \in I$. We then have that $\varphi_i : V_i \to V'$ and $\varphi_i \circ f$ is continuous. So, by Proposition 2.3.5, we have that $\varphi_i : V \to V'$ is continuous for all $i \in I$. Thus, $\iota : V \to V'$ is one of the maps over which V has the initial convergence structure. Thus, $\iota : V \to V'$ is continuous. But, tolerating a slight abuse of notation, $f : V \to W$ is exactly $f \circ \iota$ which is a composition of continuous functions, and thus continuous. QED

Corollary 4.2.14. *Let V be a CVS and $\{\varphi_i : V_i \to V\}_{i\in I}$ a family of linear maps from convergence vector spaces V_i to V. If V satisfies the property $(*)$*

> *if W is a convergence vector space and $f : V \to W$ linear, then f is continuous if and only if the composition $f \circ \varphi_i$ is continuous for all $i \in I$*

then V carries the final vector space convergence structure over $\{\varphi_i : V_i \to V\}_{i\in I}$.

Proof. Let V and V' be two CVSs satisfying $(*)$ and thus having the same underlying vector space. Let $\iota : V \to V'$ be given by $x \mapsto x$. For each $i \in I$, we have the commutative diagram



Since $\mathrm{id}_{V'} : V' \to V'$ is continuous and $\varphi_i = \mathrm{id}_{V'} \circ \varphi_i$, we have by $(*)$ that $\varphi_i : V_i \to V'$ is continuous for all $i \in I$. But then from the diagram and $(*)$, we have that ι is continuous. By identical reasoning $\iota^{-1} : V' \to V$ is continuous. The continuity of ι and ι^{-1} is exactly the statement that V and V' have the same converging filters. Thus, $V = V'$.

Since the underlying set of V with the final vector space convergence structure over $\{\varphi_i : V_i \to V\}_{i\in I}$ satisfies $(*)$, we have that if V satisfies $(*)$, then V carries the final vector space convergence structure. QED

The following theorem, following some technical lemmas, gives a more explicit description of the final vector space convergence structure.

Lemma 4.2.15. *If V is a vector space and $\mathcal{F}, \mathcal{G}$ are filters on V, then*

$$[0] \cap (\mathcal{F} + \mathcal{G}) \supseteq [0] \cap \mathcal{F} + [0] \cap \mathcal{G}.$$

Proof. Let $H \in [0] \cap \mathcal{F} + [0] \cap \mathcal{G}$. There is then $F \in \mathcal{F}$ with $0 \in F$ and $G \in \mathcal{G}$ with $0 \in G$ so that $H \supseteq F + G$. Then $0 \in F \cap G$ so that $F + G \in [0] \cap (\mathcal{F} + \mathcal{G})$ and $H \in [0] \cap (\mathcal{F} + \mathcal{G})$. This shows the desired containment. QED

Lemma 4.2.16. *If V is a vector space and $\mathcal{F}, \mathcal{G}$ are filters on V with $[0] \supseteq \mathcal{F}, \mathcal{G}$ then $\mathcal{F} \cap \mathcal{G} \supseteq \mathcal{F} + \mathcal{G}$.*

Proof. Let $H \in \mathcal{F} + \mathcal{G}$. Then there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ with $H \supseteq F + G$. Then since $0 \in F, G$, we have $F + G \supseteq G$. and $F + G \supseteq F$. Thus, $H \in \mathcal{G}$ and $H \in \mathcal{F}$. Therefore, $H \in \mathcal{F} \cap \mathcal{G}$. QED

Lemma 4.2.17. *If $\varphi : V \to W$ is a linear mapping of vector spaces, and $\mathcal{F} \in \Phi(V)$, then*

$$[0] \cap \varphi(\mathcal{F}) \supseteq \varphi([0] \cap \mathcal{F}).$$

Proof. Suppose $H \in \varphi([0] \cap \mathcal{F})$. There is then $F \in \mathcal{F}$ with $0 \in F$ so that $H \supseteq \varphi(F)$. Since $\varphi(0) = 0$ by linearity, we have $0 \in \varphi(F)$. Thus, $0 \in H$ and $H \in \varphi(\mathcal{F})$. We conclude that $H \in [0] \cap \varphi(\mathcal{F})$ as necessary for containment. QED

Theorem 4.2.18. *Let V be a vector space and $\{\varphi_i : V_i \to V \mid i \in I\}$ be a family of linear maps from convergence vector spaces V_i to V. Define Ψ to be the family of filters $\mathcal{F}$ on V so that there are finitely many indices $J \subseteq I$ and for each $j \in J$ a filter $\mathcal{F}_j$ converging to 0 in V_j as well as finitely many $v_1, ..., v_m \in V$ so that*

$$\mathcal{F} \supseteq \sum_{j \in J} \varphi_j(\mathcal{F}_j) + \sum_{k=1}^{m} \mathcal{D}v_k.$$

Then the final vector space convergence structure on V relative to the $\{\varphi_i\}_{i \in I}$ is given by $\mathcal{F} \to v$ if and only if $\mathcal{F} - v \in \Psi$.

Proof. The proof will proceed in two stages. First, we will prove that the relation $\mathcal{F} \to v$ if and only if $\mathcal{F} - v \in \Psi$ is a convergence structure by checking conditions (1)-(6) of Theorem 4.2.8. We will then prove that this relation is the final vector space convergence structure by showing that it satisfies the condition $(*)$ of Corollary 4.2.14.

We check (1), Ψ is closed under intersections. Suppose $\mathcal{F}, \mathcal{G} \in \Psi$. We then have that there are two finite subcollection $J, K \subseteq I$ and for each $j \in J$ and $k \in K$ filters $\mathcal{F}_j, \mathcal{G}_k$ converging to 0 in V_j and V_k respectively as well as finitely many $v_1, ..., v_m, w_1, ..., w_n \in V$ so that

$$\mathcal{F} \supseteq \sum_{j \in J} \varphi_j(\mathcal{F}_j) + \sum_{s=1}^{m} \mathcal{D}v_s$$

and

$$\mathcal{G} \supseteq \sum_{k \in K} \varphi_k(\mathcal{G}_k) + \sum_{t=1}^{n} \mathcal{D}w_t.$$

We then have

$$\begin{aligned}
\mathcal{F} \cap \mathcal{G} &\supseteq ([0] \cap \mathcal{F}) \cap ([0] \cap \mathcal{G}) \\
&\supseteq \left(\sum_{j \in J} [0] \cap \varphi_j(\mathcal{F}_j) + \sum_{s=1}^{m} [0] \cap \mathcal{D}v_s \right) \cap \left(\sum_{k \in K} [0] \cap \varphi_k(\mathcal{G}_k) + \sum_{t=1}^{n} [0] \cap \mathcal{D}w_t \right) \\
&\qquad \text{(Lemma 4.2.15)} \\
&\supseteq \sum_{j \in J} [0] \cap \varphi_j(\mathcal{F}_j) + \sum_{k \in K} [0] \cap \varphi_k(\mathcal{G}_k) + \sum_{s=1}^{m} [0] \cap \mathcal{D}v_s + \sum_{t=1}^{n} [0] \cap \mathcal{D}w_t \\
&\qquad \text{(Lemma 4.2.16)} \\
&= \sum_{j \in J} [0] \cap \varphi_j(\mathcal{F}_j) + \sum_{k \in K} [0] \cap \varphi_k(\mathcal{G}_k) + \sum_{s=1}^{m} \mathcal{D}v_s + \sum_{t=1}^{n} \mathcal{D}w_t \qquad (0 \in \mathbb{D}) \\
&\supseteq \sum_{j \in J} \varphi_j([0] \cap \mathcal{F}_j) + \sum_{k \in K} \varphi_k([0] \cap \mathcal{G}_k) + \sum_{s=1}^{m} \mathcal{D}v_s + \sum_{t=1}^{n} \mathcal{D}w_t. \qquad \text{(Lemma 4.2.17)}
\end{aligned}$$

Since $[0] \to 0$ in each V_i and for each $j \in J$ and $k \in K$ we have $\mathcal{F}_j \to 0$ and $\mathcal{F}_k \to 0$ we have both $[0] \cap \mathcal{F}_j \to 0$ and $[0] \cap \mathcal{F}_k \to 0$. Thus, by definition, $\mathcal{F} \cap \mathcal{G} \in \Psi$. This is condition (1) of Theorem 4.2.8.

Conditions (2) and (3) are immediate given the definition of Ψ.

We now check (4). Suppose $\mathcal{F} \in \Psi$. There are then finitely many indices $J \subseteq I$ and for each $j \in J$ a filter $\mathcal{F}_j$ converging to 0 in V_j as well as finitely many $v_1, \ldots, v_m \in V$ so that

$$\mathcal{F} \supseteq \sum_{j \in J} \varphi_j(\mathcal{F}_j) + \sum_{k=1}^{m} \mathcal{D}v_k.$$

We then have

$$\begin{aligned}
\mathcal{D}\mathcal{F} &\supseteq \sum_{j \in J} \mathcal{D}\varphi_j(\mathcal{F}_j) + \sum_{k=1}^{m} \mathcal{D}\mathcal{D}v_k && \text{(Lemma 4.2.7 (4))} \\
&= \sum_{j \in J} \mathcal{D}\varphi_j(\mathcal{F}_j) + \sum_{k=1}^{m} \mathcal{D}v_k && \text{(Lemma 4.2.7 (7))} \\
&= \sum_{j \in J} \varphi_j(\mathcal{D}\mathcal{F}_j) + \sum_{k=1}^{m} \mathcal{D}v_k && \text{(Lemma 4.2.7 (10))}
\end{aligned}$$

Since for each $j \in J$ we have that V_j is a convergence space, so $\mathcal{D}\mathcal{F}_j \to 0$. Thus, $\mathcal{D}\mathcal{F} \in \Psi$ as desired for (4).

We now check (5). Suppose $\mathcal{F} \in \Psi$. If $\alpha = 0$, then $\alpha\mathcal{F} = [0]$. Since $[0] \supseteq \mathcal{D}[0]$, we have $\alpha\mathcal{F} \in \Psi$. Else, there are finitely many indices $J \subseteq I$ and for each $j \in J$ a filter $\mathcal{F}_j$ converging to 0 in V_j as well as finitely many $v_1, ..., v_m \in V$ so that

$$\mathcal{F} \supseteq \sum_{j\in J} \varphi_j(\mathcal{F}_j) + \sum_{k=1}^{m} \mathcal{D}v_k.$$

We then have

$$\begin{aligned}
\alpha\mathcal{F} &= \sum_{j\in J} \alpha\varphi_j(\mathcal{F}_j) + \sum_{k=1}^{m} \alpha\mathcal{D}v_k && \text{(Lemma 4.2.7 (3))}\\
&= \sum_{j\in J} \alpha\varphi_j(\mathcal{F}_j) + \sum_{k=1}^{m} \mathcal{D}v_k && \text{(Lemma 4.2.7 (8))}\\
&= \sum_{j\in J} \varphi_j(\alpha\mathcal{F}_j) + \sum_{k=1}^{m} \mathcal{D}v_k && \text{(Lemma 4.2.7 (11))}
\end{aligned}$$

Since for each $j \in J$ we have that V_j is a convergence space, so $\alpha\mathcal{F}_j \to 0$ since $\mathcal{F}_j \to 0$. Thus, $\alpha\mathcal{F} \in \Psi$ as desired for (5).

We lastly have that (6) follows immediately from the definition of Ψ. We have completed the first section of the proof: the relation $\mathcal{F} \to v$ if and only if $\mathcal{F} - v \in \Psi$ is a convergence structure on V by Theorem 4.2.8.

We will now check condition ($*$) of Corollary 4.2.14. Fix $i \in I$. Suppose $\mathcal{F} \in \Phi(V_i)$ is such that $\mathcal{F} \to 0$. Then $\varphi_i(\mathcal{F}) \in \Psi$, so $\varphi_i(\mathcal{F}) \to 0$. Therefore, φ_i is continuous at 0 and by Proposition 4.1.6 we have that φ_i is continuous.

Now, fix a convergence vector space W and $f : V \to W$ linear. If f is continuous, then $f \circ \varphi_i$ is continuous for each $i \in I$.

Finally, suppose $f \circ \varphi_i$ is continuous for each $i \in I$. Suppose there is a filter $\mathcal{F} \to 0$ in V. Then $\mathcal{F} \in \Psi$ and there are finitely many indices $J \subseteq I$ and for each $j \in J$ a filter $\mathcal{F}_j$ converging to 0 in V_j as well as finitely many $v_1, ..., v_m \in V$ so that

$$\mathcal{F} \supseteq \sum_{j\in J} \varphi_j(\mathcal{F}_j) + \sum_{k=1}^{m} \mathcal{D}v_k.$$

We then have

$$f(\mathcal{F}) \supseteq \sum_{j\in J} f \circ \varphi_j(\mathcal{F}_j) + \sum_{k=1}^{m} \mathcal{D}f(v_k)$$

by Lemma 4.2.7. Each summand converges to 0 in W since each $f\circ\varphi_i$ is continuous and $\mathcal{D} \to 0$ in $\mathbb{K}$. So f is continuous at 0 and thus, by Proposition 4.1.6, we have that f is continuous.

We have verified condition ($*$) of Corollary 4.2.14 which demonstrates that the relation $\mathcal{F} \to v$ if and only if $\mathcal{F} - v \in \Psi$ is the final vector space convergence structure. QED

Corollary 4.2.19. *Let V be a CVS with the final vector space convergence structure relative to linear maps $\{\varphi_i : V_i \to V\}_{i\in I}$ out of CVSs V_i. If $V = \operatorname{span} \bigcup_{i\in I} \varphi_i(V_i)$ then a filter $\mathcal{F} \to 0$ in V if and only if we have finitely many indices $J \subseteq I$ and for each $j \in J$ a filter $\mathcal{F}_j$ converging to 0 in V_j so that*

$$\mathcal{F} \supseteq \sum_{j\in J} \varphi_j(\mathcal{F}_j).$$

Proof. Given Theorem 4.2.18, it suffices to show this containment if $\mathcal{F} \to 0$. To this end, suppose $\mathcal{F} \in \Phi(V)$ and $\mathcal{F} \to 0$. We may then find finitely many indices $J \subseteq I$ and for each $j \in J$ a filter $\mathcal{F}_j$ converging to 0 in V_j as well as finitely many $v_1, ..., v_m \in V$ so that

$$\mathcal{F} \supseteq \sum_{j\in J} \varphi_j(\mathcal{F}_j) + \sum_{k=1}^{m} \mathcal{D}v_k.$$

For each $k = 1, ..., m$ we have $v_k \in \operatorname{span} \bigcup_{i\in I} \varphi_i(V_i)$. Thus, we may find a finite subset $J_k \subseteq I$ and for each $j \in J_k$ a finite sub-collection $Z_{k,j} \subseteq V_j$ and for each $z \in Z_{k,j}$ some scalar $\alpha_{k,j}(z) \in \mathbb{K}$ so that

$$v_k = \sum_{j\in J_k} \sum_{z\in Z_{k,j}} \alpha_{k,j}(z)\varphi_j(z).$$

We then have

$$\begin{aligned} \mathcal{F} &\supseteq \sum_{j\in J} \varphi_j(\mathcal{F}_j) + \sum_{k=1}^{m} \sum_{j\in J_k} \sum_{z\in Z_{k,j}} \alpha_{k,j}(z)\mathcal{D}\varphi_j(z) \\ &= \sum_{j\in J} \varphi_j(\mathcal{F}_j) + \sum_{k=1}^{m} \sum_{j\in J_k} \sum_{z\in Z_{k,j}} \mathcal{D}\varphi_j(z) \end{aligned}$$

by Lemma 4.2.7 (8) and (13). For each $j \in J_1 \setminus J$, define $\mathcal{F}_j \in \Phi(V_j)$ by $\mathcal{F}_j = [0]$. For each $j \in J \setminus J_1$, define $Z_{1,j} = \emptyset$. Define $\widehat{J}_1 = J_1 \cup J$ and for $k = 2, ..., m$ set $\widehat{J}_k = J_k$. We then have by Lemma 4.2.7 (12) that

$$\mathcal{F} = \sum_{j\in \widehat{J}_1} \varphi_j(\mathcal{F}_j) + \sum_{k=1}^{m} \sum_{j\in \widehat{J}_k} \sum_{z\in Z_{k,j}} \mathcal{D}\varphi_j(z).$$

Define $\mathcal{F}_{1,j} = \mathcal{F}_j$. For each $k = 2, ..., m$ and $j \in \widehat{J}_k$, define $\mathcal{F}_{k,j} \in \Phi(V_j)$ by $\mathcal{F}_{k,j} = [0]$.

Thus,

$$\begin{aligned}
\mathcal{F} &= \sum_{k=1}^{m} \sum_{j \in \widehat{J}_k} \varphi_j(\mathcal{F}_{k,j}) + \sum_{k=1}^{m} \sum_{j \in \widehat{J}_k} \sum_{z \in Z_{k,j}} \mathcal{D}\varphi_j(z) && \text{(Lemma 4.2.7 (12))} \\
&= \sum_{k=1}^{m} \sum_{j \in \widehat{J}_k} \left(\varphi_j(\mathcal{F}_{k,j}) + \sum_{z \in Z_{k,j}} \mathcal{D}\varphi_j(z) \right) \\
&= \sum_{k=1}^{m} \sum_{j \in \widehat{J}_k} \left(\varphi_j(\mathcal{F}_{k,j}) + \varphi_j\left(\sum_{z \in Z_{k,j}} \mathcal{D}z \right) \right) \\
&= \sum_{k=1}^{m} \sum_{j \in \widehat{J}_k} \varphi_j\left(\mathcal{F}_{k,j} + \sum_{z \in Z_{k,j}} \mathcal{D}z \right).
\end{aligned}$$

Since

$$\mathcal{F}_{k,j} + \sum_{z \in Z_{k,j}} \mathcal{D}z \to 0$$

we have the desired result (technically after reordering the sum to group like indices from I and pulling φ_i over the sum once more). QED

Proposition 4.2.20. *The quotient convergence structure on a CVS and the final vector space convergence structure relative to the quotient map coincide.*

Proof. Let $q : V \to W$ be a linear surjection with V a CVS. Let W_q denote W with its quotient convergence structure and W_f denote W with the final vector space convergence structure relative to the quotient map. Suppose we have a filter $\mathcal{F} \to w$ in W_q. Recall from Lemma 4.1.10 that this means we have a single $v \in q^{-1}(w)$ and filter $\mathcal{G} \to v$ so that $\mathcal{F} \supseteq q(\mathcal{G})$. An immediate consequence is that

$$\begin{aligned}
\mathcal{F} - w &\supseteq q(\mathcal{G}) - q(v) \\
&= q(\mathcal{G} - v)
\end{aligned}$$

where of course $\mathcal{G} - v \to 0$. We thus have that $\mathcal{F} - w \to 0$ in W_f by continuity of q. We conclude that $\mathcal{F} \to w$ in W_f.

For the other direction, let $\mathcal{F} \to w$ in W_f. We then have that $\mathcal{F} - w \to 0$. By Corollary 4.2.19 we may produce a filter $\mathcal{F} \to 0$ in V so that

$$\mathcal{F} - w \supseteq q(\mathcal{F}).$$

Letting $v \in V$ with $q(v) = w$, we recover that

$$\mathcal{F} \supseteq q(\mathcal{F} + v)$$

with $\mathcal{F} + v \to v$ in V. This is precisely that $\mathcal{F} \to w$ in W_q by Lemma 4.1.10. QED

Corollary 4.2.21. *Quotients of topological CVSs are topological.*

Definition 4.2.22. If $\{V_i\}_{i\in I}$ is a family of convergence spaces, their *coproduct* is defined to be $\bigoplus_{i\in I} V_i$ endowed with its final vector space convergence structure relative to the usual embeddings $e_i : V_i \to \bigoplus_{i\in I} V_i$. Unless otherwise stated, $\bigoplus_{i\in I} V_i$ will always be assumed to carry this convergence structure.

Remark 4.2.23. Similarly to the convergence sum, this is clearly the coproduct in the category of convergence $\mathbb{K}$-vector spaces.

The last result of this section will be a classification of all finite dimensional Hausdorff convergence vector spaces.

Lemma 4.2.24. *A filter $\mathcal{F}$ on $\mathbb{K}^n$ (with its usual topology) converges to v if and only if*

$$\mathcal{F} \supseteq \sum_{i=1}^{n} \mathcal{D}e_i + v$$

where e_i is the i-th standard basis vector.

Proof. Since $\mathbb{K}^n$ is topological, this amounts to proving that for all $v \in \mathbb{K}^n$ the neighborhood filter for v is

$$\sum_{i=1}^{n} \mathcal{D}e_i + v.$$

Suppose U is a neighborhood of $v \in V$. We have that $U \supseteq B_\epsilon(v)$ for some $\epsilon > 0$. It is then not hard to see that

$$B_\epsilon(v) \supseteq \sum_{i=1}^{n} \frac{\epsilon}{n}\mathbb{D}e_i + v$$

so that

$$U \in \sum_{i=1}^{n} \mathcal{D}e_i + v$$

and

$$\mathcal{N}_v \subseteq \sum_{i=1}^{n} \mathcal{D}e_i + v.$$

Similarly, if $\epsilon_1, ..., \epsilon_n > 0$, it is not hard to check

$$\sum_{i=1}^{n} (-\epsilon_i, \epsilon_i)e_i + v \supseteq B_\epsilon(v).$$

where

$$\epsilon = \min_{1 \leq i \leq n} \epsilon_i.$$

Thus,

$$\sum_{i=1}^{n} (-\epsilon_i, \epsilon_i) e_i + v \in \mathcal{N}_v,$$

and

$$\mathcal{N}_v \supseteq \sum_{i=1}^{n} \mathcal{D} e_i + v.$$

We conclude

$$\mathcal{N}_v = \sum_{i=1}^{n} \mathcal{D} e_i + v$$

as desired. QED

Theorem 4.2.25. *If V is an n-dimensional, Hausdorff CVS, then V is isomorphic to $\mathbb{K}^n$. Further, any bijection $f : K^n \to V$ is a homeomorphism.*

Proof. Let $f : \mathbb{K}^n \to V$ be a linear bijection. For each $i = 1, ..., n$ let e_i denote the i-th standard basis vector for $\mathbb{K}^n$ and set $v_i = f(e_i)$. Since V is n-dimensional, we have that $\{v_1, ..., v_n\}$ is a basis for V.

Suppose that $v \in \mathbb{K}^n$ and that there is a filter $\mathcal{F} \to v$. By Lemma 4.2.24 we have that

$$\mathcal{F} \supseteq \sum_{i=1}^{n} \mathcal{D} e_i + v.$$

Therefore,

$$f(\mathcal{F}) \supseteq \sum_{i=1}^{n} \mathcal{D} v_i + f(v) \to f(v).$$

Therefore, $f(\mathcal{F}) \to f(v)$ and we have that f is continuous.

Let $\overline{\mathbb{D}}^n$ and S^n denote the closed unit ball and unit sphere in $\mathbb{K}^n$ respectively. We know that S^n is compact and $f(S^n)$ is compact and closed by continuity of f (Proposition 2.5.16) and Hausdorffness of V (Proposition 2.5.14).

Suppose we have some filter $\mathcal{F} \to 0$ in V. We know that $\mathcal{D}\mathcal{F} \to 0$. We claim that we may find $F \in \mathcal{F}$ and $r > 0$ so that $r\mathbb{D}F \cap f(S^n) = \emptyset$. This must be true since otherwise, we could consider the filter

$$\mathcal{G} = [\{U \cap f(S^n) : U \in \mathcal{DF}\}]$$

Certainly, $\mathcal{G} \supseteq \mathcal{DF}$ and $f(S^n) \in \mathcal{G}$. Thus, $\mathcal{G} \to 0$ and $0 \in \alpha(f(S^n)) = f(S^n)$. This is impossible since f is a bijection.

Now, choose $F \in \mathcal{F}$ and $r > 0$ so that $r\mathbb{D}F \cap f(S^n) = \emptyset$ and define $K = f(\frac{1}{r}\overline{\mathbb{D}}^n)$. Since K is the continuous image of a compact set, K is compact by Proposition 2.5.16. We claim that $F \subseteq K$. Otherwise, suppose there is $v \in F$ with $v \notin K$. Defining $w = f^{-1}(v)$, we have $w \notin \frac{1}{r}\overline{\mathbb{D}}^n$ and $|w| > 1/r$. We then have $|rw| > 1$ and there is some $\lambda \in \mathbb{K}$ with $|\lambda| < 1$ so that $\lambda rw \in S^n$. Then $r\lambda v \in f(S^n)$. Since $v \in F$ and $|\lambda| < 1$, we have $r\lambda v \in r\mathbb{D}F$. This is a contradiction since $r\mathbb{D}F \cap f(S^n) = \emptyset$.

Let $g : \frac{1}{r}\mathbb{D}^n \to K$ be the restriction of f to this domain. We have that g is a continuous bijection from a compact Choquet space to a Hausdorff space and is thus a homeomorphism by Proposition 2.6.36. Let $\iota : K \to V$ be the inclusion. Consider the filter on K given by

$$\mathcal{F}|_F = [\{U \cap F : U \in \mathcal{F}\}].$$

Note that if $U \in \mathcal{F}$, then $U \supseteq F \cap U \in \iota(\mathcal{F}|_F)$. Thus, $\iota(\mathcal{F}|_F) \supseteq \mathcal{F}$ and $\iota(\mathcal{F}|_F) \to 0$. We conclude that $\mathcal{F}|_F \to 0$ in the subspace convergence structure on K. Thus, by continuity of g, we have $g^{-1}(\mathcal{F}|_F) \to 0$ in $\mathbb{K}^n$.

Let $H \in g^{-1}(\mathcal{F}_F)$. There is then $U \in \mathcal{F}$ so that $H \supseteq g^{-1}(U \cap F)$. Since $U \cap F \subseteq K$, we have $H \supseteq f^{-1}(U \cap F)$. Since $F \in \mathcal{F}$, we have $H \in f^{-1}(\mathcal{F})$. Thus,

$$f^{-1}(\mathcal{F}) \supseteq g^{-1}(\mathcal{F}|_F)$$

we have $f^{-1}(\mathcal{F}) \to 0$ so that f^{-1} is continuous. QED

Corollary 4.2.26. *If V and W are finite dimensional Hausdorff CVSs, then any linear map $f : V \to W$ is continuous.*

Proof. Let $f : V \to W$ be a linear map. We have some linear homeomorphisms $\varphi : V \to \mathbb{K}^n$ and $\psi : W \to \mathbb{K}^m$ for $n = \dim V$ and $M = \dim W$. This gives rise to the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \varphi\downarrow & & \uparrow\psi^{-1} \\ \mathbb{K}^n & \xrightarrow[\psi f \varphi^{-1}]{} & \mathbb{K}^m \end{array}$$

We know $\psi f \varphi^{-1}$ is continuous since it is a linear map $\mathbb{K}^n \to \mathbb{K}^m$. We conclude that f is continuous as a composition of linear maps. QED

Corollary 4.2.27. *If V and W are finite dimensional Hausdorff CVSs, then any linear bijection $V \to W$ is a homeomorphism.*

Corollary 4.2.28. *A linear map from a CVS to $\mathbb{K}$ is continuous if and only if its kernel is closed.*

Proof. Fix a CVS V and a linear functional $\varphi : V \to \mathbb{K}$.

Suppose φ is continuous. Let $v \in a(\ker\varphi)$. We then have a net $\alpha \to v$ with α in $\ker\varphi$. Then $\varphi(\alpha) \to \varphi(v)$. But $\varphi(\alpha)$ is merely the constant zero net in $\mathbb{K}$ which is Hausdorff. We then have that $\varphi(v) = 0$ and $v \in \ker\varphi$. Therefore, the kernel is closed.

Now, suppose that $\ker\varphi$ is closed. We have that $V/\ker\varphi$ is Hausdorff by Proposition 4.1.13. Further φ descends to the quotient, giving an injective map $\widetilde{\varphi} : V/\ker\varphi \to \mathbb{K}$. Since $V/\ker\varphi$ is finite dimensional (specifically at most one dimensional), we have that $\widetilde{\varphi}$ is continuous. Therefore $\varphi = \widetilde{\varphi} \circ q$ is continuous with $q : V \to V/\ker\varphi$ the quotient map. QED

4.3 Locally Convex Convergence Vector Spaces

In this section we will consider locally convex convergence vector spaces, the convergence analogue to locally convex topological vector spaces. Before giving the definition (Definition 4.3.7) we will define the convex and absolutely convex hulls of a filter and prove some basic properties.

Definition 4.3.1. If V is a vector space, we define the *convex hull* of a filter $\mathcal{F}$ on V by

$$\mathrm{co}(\mathcal{F}) = [\{\mathrm{co}(F) : F \in \mathcal{F}\}],$$

that is, the filter generated by the convex hulls of the elements of $\mathcal{F}$. Likewise, we define the *absolute convex hull* of $\mathcal{F}$ by

$$\Gamma(\mathcal{F}) = [\{\Gamma(F) : F \in \mathcal{F}\}],$$

that is, the filter generated by the absolute convex hulls of the elements of $\mathcal{F}$.

Proposition 4.3.2. *Fix a vector space V and a filter $\mathcal{F}$ on V. If $H \in \mathrm{co}(\mathcal{F})$, there is $F \in \mathcal{F}$ with $H \supseteq \mathrm{co}(F)$. If $H \in \Gamma(\mathcal{F})$, there is $F \in \mathcal{F}$ with $H \supseteq \Gamma(F)$.*

Proof. Suppose $H \in \mathrm{co}(\mathcal{F})$. There are then $F_1, ..., F_n \in \mathcal{F}$ with

$$\begin{aligned} H &\supseteq \bigcap_{i=1}^{n} \mathrm{co}(F_n) \\ &\supseteq \mathrm{co}\left(\bigcap_{i=1}^{n} F_n\right). \end{aligned}$$

Since $\bigcap_{i=1}^{n} F_n \in \mathcal{F}$, the result follows.

Next, suppose $H \in \Gamma(\mathcal{F})$. There are then $F_1, ..., F_n \in \mathcal{F}$ with

$$\begin{aligned} H &\supseteq \bigcap_{i=1}^{n} \Gamma(F_n) \\ &\supseteq \Gamma\left(\bigcap_{i=1}^{n} F_n\right). \end{aligned}$$

Since $\bigcap_{i=1}^{n} F_n \in \mathcal{F}$, the result follows. QED

Proposition 4.3.3. *If V is a vector space and $\mathcal{F} \in \Phi(V)$, then*

$$\mathcal{F} \supseteq \mathrm{co}(\mathcal{F}) \supseteq \Gamma(\mathcal{F}).$$

Proof. Suppose $H \in \Gamma(\mathcal{F})$. There is then some $F \in \mathcal{F}$ so that $H \supseteq \Gamma(F)$. Since $\Gamma(F)$ is convex and contains F, we have $\Gamma(F) \supseteq \mathrm{co}(F)$. Therefore, $H \in \mathrm{co}(\mathcal{F})$ and $\mathrm{co}(\mathcal{F}) \supseteq \Gamma(\mathcal{F})$.

Suppose now $H \in \mathrm{co}(\mathcal{F})$. There is then some $F \in \mathcal{F}$ so that $H \supseteq \mathrm{co}(F) \supseteq F$. Therefore, $H \in \mathcal{F}$ and $\mathcal{F} \supseteq \mathrm{co}(\mathcal{F})$. QED

Proposition 4.3.4. *If V is a vector space and $\mathcal{F}, \mathcal{G} \in \Phi(V)$ with $\mathcal{F} \supseteq \mathcal{G}$, then $\mathrm{co}(\mathcal{F}) \supseteq \mathrm{co}(\mathcal{G})$ and $\Gamma(\mathcal{F}) \supseteq \Gamma(\mathcal{G})$.*

Proof. Suppose $H \in \mathrm{co}(\mathcal{G})$. There is then $G \in \mathcal{G}$ with $H \supseteq \mathrm{co}(G)$. But since $G \in \mathcal{F}$, we have $H \in \mathrm{co}(\mathcal{F}$. Thus, $\mathrm{co}(\mathcal{F}) \supseteq \mathrm{co}(\mathcal{G})$. The proof of $\Gamma(\mathcal{F}) \supseteq \Gamma(\mathcal{G})$ is identical. QED

Proposition 4.3.5. *If V and W are vector spaces, $f : V \to W$ linear, and $\mathcal{F}$ a filter on V, then $f(\mathrm{co}(\mathcal{F})) = \mathrm{co}(f(\mathcal{F}))$ and $f(\Gamma(\mathcal{F})) = \Gamma(f(\mathcal{F}))$.*

Proof. Suppose $H \in f(\mathrm{co}(\mathcal{F}))$ is equivalent to the existence of $F \in \mathcal{F}$ with $H \supseteq f(\mathrm{co}(F))$. Since $f(\mathrm{co}(F)) = \mathrm{co}(f(F))$, this is equivalent to $H \in \mathrm{co}(\mathcal{F}))$. Thus, $f(\mathrm{co}(\mathcal{F})) = \mathrm{co}(f(\mathcal{F}))$. The proof of $f(\Gamma(\mathcal{F})) = \Gamma(f(\mathcal{F}))$ is identical. QED

Proposition 4.3.6. *If V is a vector space and $\mathcal{F}, \mathcal{G}$ are filters on V, then $\mathrm{co}(\mathcal{F} + \mathcal{G}) = \mathrm{co}(\mathcal{F}) + \mathrm{co}(\mathcal{G})$ and $\Gamma(\mathcal{F} + \mathcal{G}) = \Gamma(\mathcal{F}) + \Gamma(\mathcal{G})$*

Proof. Suppose $H \in \mathrm{co}(\mathcal{F} + \mathcal{G})$. There are then $F \in \mathcal{F}$ and $G \in \mathcal{G}$ so that $H \supseteq \mathrm{co}(F+G)$. Since $\mathrm{co}(F+G) = \mathrm{co}(F)+\mathrm{co}(G)$, this is equivalent to $H \in \mathrm{co}(\mathcal{F})+\mathrm{co}(\mathcal{G})$. Thus, $\mathrm{co}(\mathcal{F} + \mathcal{G}) = \mathrm{co}(\mathcal{F}) + \mathrm{co}(\mathcal{G})$.

The proof for the absolute convex hull is identical. QED

We can now define locally convex convergence vector spaces.

Definition 4.3.7. A CVS is called *locally convex* if whenever a filter $\mathcal{F} \to 0$ we have that $\mathrm{co}(\mathcal{F}) \to 0$.

We next justify that this definition does generalize that of locally convex topological vector spaces.

Proposition 4.3.8. *A TVS V is locally convex if and only $\mathfrak{C}(V)$ is a locally convex CVS.*

Proof. Let V be a locally convex TVS. Recalling that $\mathcal{N}_0$ denotes the neighborhood filter at 0 in V, we have that $\operatorname{co}(\mathcal{N}_0) \subseteq \mathcal{N}_0$ and since V is locally convex, we have that $\operatorname{co}(\mathcal{N}_0) = \mathcal{N}_0$. Now, if $\mathcal{F} \to 0$ in $\mathfrak{C}(V)$ we have that $\mathcal{F} \supseteq \mathcal{N}_0$. Then we have that $\operatorname{co}(\mathcal{F}) \supseteq \operatorname{co}(\mathcal{N}_0) = \mathcal{N}_0$ so that $\operatorname{co}(\mathcal{F}) \to 0$. Thus, $\mathfrak{C}(V)$ is a locally convex CVS.

Now, suppose that V is a TVS and $\mathfrak{C}(V)$ is locally convex. Since $\mathcal{N}_0 \to 0$, we have that $\operatorname{co}(\mathcal{N}_0) \to 0$, but then $\mathcal{N}_0 \subseteq \operatorname{co}(\mathcal{N}_0)$. Thus, any neighborhood N of 0 in V is an element of $\operatorname{co}(\mathcal{N}_0)$. But then $N \supseteq \operatorname{co}(N')$ for some $N' \in \mathcal{N}_0)$ so that N contains a convex neighborhood of 0. This means that V is locally convex as a TVS. QED

Proposition 4.3.9. *If V is a locally convex CVS than so are its subspaces.*

Proof. Let U be a subspace of V and $\iota : U \to V$ the embedding. Suppose $\mathcal{F} \to 0$ in U. Then $\iota(\mathcal{F}) \to 0$ in V. So we have that $\operatorname{co}(\iota(\mathcal{F}) \to 0$ in V. Since $\operatorname{co}(\iota(\mathcal{F}) = \iota(\operatorname{co}\mathcal{F})$, one has $\operatorname{co}(\mathcal{F}) \to 0$ in U. Thus, U is locally convex. QED

Proposition 4.3.10. *If V is a locally convex CVS then so are its quotients.*

Proof. Suppose W is a quotient of V with quotient map q. Suppose $\mathcal{F} \to 0$ in W. By Lemma 4.1.10, there is some filter $\mathcal{G}$ on V which converges to $x \in \ker q$ so that $\mathcal{F} \supseteq q(\mathcal{G})$. We then have that $\mathcal{G} - x \to 0$. It follows that $\operatorname{co}(\mathcal{G} - x) \to 0$. By continuity of q we have that $q(\operatorname{co}(\mathcal{G} - x)) \to 0$ in W. We then see that

$$\begin{aligned} \operatorname{co}(\mathcal{F}) &\supseteq \operatorname{co}(q(\mathcal{G})) \\ &= \operatorname{co}(q(\mathcal{G} - x)) \\ &= q(\operatorname{co}(\mathcal{G} - x)) \\ &\to 0 \end{aligned}$$

so $\operatorname{co}(\mathcal{F}) \to 0$ in W. QED

Proposition 4.3.11. *Products of locally convex CVSs are locally convex.*

Proof. Let $\{V_i\}_{i \in I}$ be a family of locally convex CVSs and let $V = \prod_{i \in I} V_i$. Suppose $\mathcal{F} \to 0$ in V. Then for each $i \in I$ we have that $\pi_i(\mathcal{F}) \to 0$ in V_i. Since each V_i is locally convex, we have that

$$\operatorname{co}(\pi_i(\mathcal{F})) = \pi_i(\operatorname{co}(\mathcal{F})) \to 0$$

in V_i. As this holds for all $i \in I$, we have that $\operatorname{co}(\mathcal{F}) \to 0$ in V. Therefore, V is locally convex. QED

Proposition 4.3.12. *Direct sums of locally convex CVSs are locally convex.*

Proof. Let $\{V_i\}_{i\in I}$ be a family of locally convex CVSs and let $V = \bigoplus_{i\in I} V_i$. Suppose $\mathcal{F} \to 0$ in V. We then have by Corollary 4.2.19 a finite set $J \subseteq I$ and filter $\mathcal{F}_j \to 0$ in X_j for each $j \in J$ so that

$$\mathcal{F} \supseteq \sum_{j\in J} \varphi_j(\mathcal{F}_j).$$

Of course then

$$\mathrm{co}(\mathcal{F}) \supseteq \sum_{j\in J} \varphi_j(\mathrm{co}(\mathcal{F}_j)).$$

and since each $\mathrm{co}(\mathcal{F}_j) \to 0$ in V_j by local convexity we have $\mathrm{co}(\mathcal{F}) \to 0$ in V. QED

Lemma 4.3.13. *If V, W are (convergence) vector spaces, then $\mathrm{ev} : \mathcal{L}(V,W) \times V \to W$ is bilinear.*

Proof. Fix $v \in V$. Suppose $f, g \in \mathcal{L}(V,W)$ and $\alpha \in \mathbb{K}$. Then

$$\begin{aligned} \mathrm{ev}(f+\alpha g, v) &= (f+\alpha g)(v) \\ &= f(v) + \alpha g(v) \\ &= \mathrm{ev}(f,v) + \alpha\,\mathrm{ev}(g,v). \end{aligned}$$

Fix $f \in \mathcal{L}(V,W)$. Suppose $v, w \in V$ and $\alpha \in \mathbb{K}$. Then

$$\begin{aligned} \mathrm{ev}(f, v+\alpha w) &= f(v+\alpha w) \\ &= f(v) + \alpha f(w) \\ &= \mathrm{ev}(f,v) + \alpha\,\mathrm{ev}(f,w). \end{aligned}$$

QED

Lemma 4.3.14. *Suppose V and W are CVSs, $\mathscr{F}$ is a filter on $\mathcal{L}(V,W)$, $\mathcal{F}$ is a filter on V, then*

$$\mathrm{ev}(\mathrm{co}(\mathscr{F}) \times \mathcal{F}) \supseteq \mathrm{co}(\mathrm{ev}(\mathscr{F} \times \mathcal{F})).$$

Proof. This is an immediate consequence of Lemma 4.3.13 and Proposition D.1.7. QED

Proposition 4.3.15. *If V and W are CVSs and W is locally convex, then $\mathcal{L}(V,W)$ is locally convex.*

Proof. Suppose $\mathscr{F} \to 0$ in $\mathcal{L}(V,W)$. We then have for any $v \in V$ and filter $\mathcal{F} \to v$ that $\mathrm{ev}(\mathscr{F} \times \mathcal{F}) \to 0$ in W. We then have that $\mathrm{co}(\mathrm{ev}(\mathscr{F} \times \mathcal{F})) \to 0$ by local convexity of W. By Lemma 4.3.14, we then have that $\mathrm{ev}(\mathrm{co}(\mathscr{F}) \times \mathcal{F}) \to 0$. This shows that $\mathrm{co}(\mathscr{F}) \to 0$ in $\mathcal{L}(V,W)$. QED

4.4 Locally Convex Topological Modification

In this section, we will describe a method for turning any convergence vector space into a locally convex topological vector space. This will give rise to a functor and natural transformation sharing many of the same properties as a strict modification of convergence spaces.

Recall that if V is a vector space over $\mathbb{K}$, a seminorm is a function $p : V \to \mathbb{R}$ so that for all $v, w \in V$ and $\lambda \in \mathbb{K}$ we have

1. $p(v + w) \leq p(v) + p(w)$;
2. $p(\lambda v) = |\lambda| p(v)$.

Note that a seminorm p must be non-negative since for any $v \in V$ one has

$$\begin{aligned} 0 &= p(0) \\ &= p(v - v) \\ &\leq p(v) + p(-v) \\ &= 2p(v). \end{aligned}$$

Definition 4.4.1. If V is a CVS, denote by $S(V)$ the set of continuous seminorms $p : V \to \mathbb{R}$. For each $w \in V$ and $p \in S(V)$, define $p_w : V \to \mathbb{R}$ by $p_w(v) = p(v - w)$. Define the *locally convex topological modification of* V to be $\mathfrak{T}(W)$ where W is the underlying set of V equipped with the initial convergence structure relative to

$$S_V := \{p_w : V \to \mathbb{R} \mid p \in S(V) \quad \text{and} \quad w \in V\}.$$

The resulting TVS is denoted $\ell(V)$.

Remark 4.4.2. Since $\mathbb{R}$ is topological, we have that W is topological. Thus we have that $\mathfrak{C}\mathfrak{T}(W) = W$. This means that $\mathfrak{C}\ell(V)$ is V with the initial convergence structure relative to S_V.

Proposition 4.4.3. *If V is a CVS, a net α in $\mathfrak{C}\ell(V)$ converges to $v \in \mathfrak{C}\ell(V)$ if and only if for every continuous seminorm $p : V \to \mathbb{R}$ one has $p(\alpha - v) \to 0$.*

Proof. Given the definition of $\mathfrak{C}\ell(V)$, it suffices to prove sufficiency. Suppose α is a net in $\mathfrak{C}\ell(V)$ so that for every continuous seminorm $p : V \to \mathbb{R}$ one has $p(\alpha - v) \to 0$. Fix $p \in S(V)$ and $w \in V$. Since

$$|p(\alpha - w) - p(v - w)| \leq p(\alpha - v)$$

by the reverse triangle inequality, one has $p(\alpha - w) \to p(v - w)$. Thus, $\alpha \to v$. QED

Right now, all we know is that $\ell(V)$ is a topological space and a vector space. We want $\ell(V)$ to be a locally convex topological vector space. This will be established by the next few propositions.

Proposition 4.4.4. *If V is a CVS, then $\ell(V)$ is a TVS.*

Proof. It suffices to check that $\mathfrak{C}\ell(V)$ is a CVS.

Suppose α and β are nets in $\mathfrak{C}\ell(V)$ with $\alpha \to v$ and $\beta \to u$ for some $v, u \in \mathfrak{C}\ell(V)$. Fix a continuous seminorm $p \in S(V)$. Since $\mathfrak{C}\ell(V)$ carries the initial convergence over S_V, we have that $p(\alpha - v) \to 0$ and $p(\beta - u) \to 0$. Thus, $p(\alpha + \beta - (v + u)) \to 0$ since

$$p(\alpha + \beta - (v + u)) \leq p(\alpha - v) + p(\beta - v).$$

Thus, by Proposition 4.4.3, we have that $\alpha + \beta \to v + u$. Therefore, vector addition $+ : \mathfrak{C}\ell(V) \times \mathfrak{C}\ell(V) \to \mathfrak{C}\ell(V)$ is continuous.

Now, fix nets δ in $\mathbb{K}$ and α in $\mathfrak{C}\ell(V)$ so that $\delta \to \lambda$ for some scalar $\lambda \in \mathbb{K}$ and $\alpha \to v$ for some $v \in \mathfrak{C}\ell(V)$. Fix continuous seminorm $p \in S(V)$. Let $\epsilon > 0$.

Since $\delta \to \lambda$, we may find $i_0 \in \operatorname{dom} \delta$ so that for all $i \geq i_0$ one has

$$\begin{cases} |\delta_i - \lambda| < \epsilon/(2p(v)) & p(v) \neq 0 \\ |\delta_i - \lambda| < 1 & p(v) = 0 \end{cases}$$

Note that for $i \geq i_0$ on must also have $|\delta_i| < M$ for some $M > 0$. Since $\alpha \to v$, one has $p(\alpha - v) \to 0$. Thus, there is some $j_0 \in \operatorname{dom} \alpha$ so that for all $j \in \operatorname{dom} \alpha$ with $j \geq j_0$ one has $p(\alpha_j - v) < \epsilon/(2M)$.

With this setup, one has for all $(i, j) \geq (i_0, j_0)$, and so

$$\begin{aligned} p(\delta_i \alpha_j - \lambda v) &= p(\delta_i \alpha_j - \delta_i v + \delta_i v - \lambda v) \\ &\leq p(\delta_i \alpha_j - \delta_i v) + p(\delta_i v - \lambda v) \\ &= |\delta_i| p(\alpha_j - v) + |\delta_i - \lambda| p(v) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Thus, $p(\delta\alpha - \lambda v) \to 0$. As this holds for all $p \in S(V)$, we have that $\delta\alpha \to \lambda v$ by Proposition 4.4.3. Thus, scalar multiplication $\cdot : \mathbb{K} \times \mathfrak{C}\ell(V) \to \mathfrak{C}\ell(V)$ is continuous. We may now conclude that $\mathfrak{C}\ell(V)$ is a CVS and $\ell(V)$ is a TVS. QED

Proposition 4.4.5. *If V is a CVS, then $\ell(V)$ is locally convex.*

Proof. For each continuous seminorm $p : V \to \mathbb{R}$ and $w \in \ell(V)$ and $\epsilon > 0$, define $B_p(w, \epsilon) \subseteq \ell(V)$ by

$$\begin{aligned} B_p(w, \epsilon) &= \{v \in \ell(V) : p(v - w) < \epsilon\} \\ &= p_w^{-1}(-\epsilon, \epsilon). \end{aligned}$$

Since each $B_p(w,\epsilon)$ is the preimage of an open set under a continuous map, we have that each $B_p(w,\epsilon)$ is open. Further, note that if $u, v \in B_p(w,\epsilon)$ and $\lambda \in [0,1]$, then

$$\begin{aligned} p(\lambda v + (1-\lambda)u - w) &= p(\lambda v + (1-\lambda)u - \lambda w - (1-\lambda)w) \\ &= p(\lambda(v-w) + (1-\lambda)(u-w)) \\ &\leq \lambda p(v-w) + (1-\lambda)p(u-w) \\ &< \epsilon \end{aligned}$$

so that $B_p(w,\epsilon)$ is convex.

Since $\ell(V)$ carries the initial topology over S_V, a generic sub-basic open set of $\ell(V)$ containing 0 is of the form $p_w^{-1}(U)$ for $p \in S(V)$ and $w \in V$ and $U \subseteq \mathbb{R}$ open with $0 \in p_w^{-1}(U)$. Since $0 \in p_w^{-1}(U)$, one has that $p(w) \in U$. Since U is open, there is some $\epsilon > 0$ so that $(p(w)-\epsilon, p(w)+\epsilon) \subseteq U$. Suppose $v \in B_p(0,\epsilon)$. Then

$$\begin{aligned} |p_w(v) - p(w)| &= |p(v-w) - p(-w)| \\ &\leq p(v-w+w) \\ &= p(v) \\ &< \epsilon \end{aligned}$$

and $p_w(v) \in U$. Thus,

$$0 \in B_p(0,\epsilon) \subseteq p_w^{-1}(U).$$

Thus, any open set in $\ell(V)$ containing 0 contains an intersection of balls of the form $B_p(0,\epsilon)$ which is convex. Therefore, $\ell(V)$ has a neighborhood base at 0 consisting of convex sets. QED

We have now shown that if V is a convergence vector space, then $\mathfrak{C}\ell(V)$ is a locally convex convergence vector space.

Definition 4.4.6. If V is a CVS, define $\iota_V : V \to \mathfrak{C}\ell(V)$ by $x \mapsto x$. If V, W are CVSs and $\varphi : V \to W$ is a linear map, define $\mathfrak{C}\ell(\varphi) : \mathfrak{C}\ell(V) \to \mathfrak{C}\ell(W)$ by $x \mapsto \varphi(x)$.

Denote by $\mathbf{cvs}_{\mathbb{K}}$ the categories of convergence vector spaces over $\mathbb{K}$ and $\mathbb{K}$-linear maps. The next result establishes that $\mathfrak{C}\ell : \mathbf{cvs}_{\mathbb{K}} \to \mathbf{cvs}_{\mathbb{K}}$ is a functor and $\iota :$ $\mathrm{id}_{\mathbf{cvs}} \to \mathfrak{C}\ell$ is a natural transformation.

Theorem 4.4.7. *Suppose V, W are CVSs.*

1. *The map $\iota_V : V \to \mathfrak{C}\ell(V)$ is continuous and a homeomorphism if and only if V is locally convex and topological.*

2. *If $\varphi : V \to W$ is continuous and linear, then $\mathfrak{C}\ell(\varphi) : \mathfrak{C}\ell(V) \to \mathfrak{C}\ell(W)$ is continuous.*

3. *If U is a locally convex topological CVS and $\varphi : V \to U$ is linear, then the following are equivalent.*

   (i) φ *is continuous.*

   (ii) $\mathfrak{Cl}(\varphi) : \mathfrak{Cl}(V) \to \mathfrak{Cl}(U)$ *is continuous.*

Proof. Fix convergence vector spaces V and W.

1. If $p : V \to \mathbb{R}$ is a continuous seminorm and $w \in V$, then $p_w : V \to \mathbb{R}$ is continuous and

$$\begin{array}{ccc} V & \xrightarrow{\iota_V} & \mathfrak{Cl}(V) \\ & \searrow^{p_w} & \downarrow p_w \\ & & \mathbb{R} \end{array}$$

   commutes. Thus, by the universal property of the initial convergence structure, i_V is continuous. Further, if ι_V is a homeomorphism, then V and $\mathfrak{Cl}(V)$ have the same converging filters and so V is locally convex and topological.

   Suppose V is locally convex and topological. Suppose α is a net in $\mathfrak{Cl}(V)$ with $\alpha \to 0$ in $\mathfrak{Cl}(V)$. Then by Proposition 4.4.3 we have for all continuous seminorms $p : V \to \mathbb{R}$ we have $p(\alpha) \to 0$. Suppose A is a neighborhood of 0 in V. By Proposition D.3.2, we may assume A is absolutely convex. Then by Proposition D.2.4 and Proposition D.2.7 we have that the Minkowski seminorm $p_A : V \to \mathbb{R}$ given by

$$p_A(v) = \inf\{r > 0 : v \in rA\}$$

   is continuous. Thus, $p_A(\alpha) \to 0$. Thus, there is some $i_0 \in \operatorname{dom} \alpha$ so that for all $i \in \operatorname{dom} \alpha$ with $i \geq i_0$ we have $p_A(\alpha_i) < 1$ so that $\alpha_i \in rA$ for some $1 > r > 0$. Since A is balanced, $\alpha_i \in U$ and $\alpha_i \in_{\text{ev}} A$. We conclude that $\alpha \to 0$ in V. Since $i_V^{-1} : \mathfrak{Cl}(V) \to V$ is linear and we have just shown it to be continuous at 0, we have i_V^{-1} is continuous. Thus, i_V is a homeomorphism.

2. Suppose α is a net $\mathfrak{Cl}(V)$ converging to 0. Suppose $p : W \to \mathbb{R}$ is a continuous seminorm. Then since φ is continuous, $p \circ \varphi : V \to \mathbb{R}$ is a continuous seminorm. So, by Proposition 4.4.3, we have $p \circ \varphi(\alpha) \to 0$. Thus, $p(\varphi(\alpha)) \to 0$ and $\varphi(\alpha) \to 0$ in $\mathfrak{Cl}(V)$ by Proposition 4.4.3. Therefore, $\mathfrak{Cl}(\varphi)$ is continuous at 0 and so continuous everywhere.

3. Note that since U is locally convex and topological, ι_U is a homeomorphism by (1). If φ is continuous, then $\mathfrak{Cl}(\varphi)$ is continuous by (2). If on the other hand $\mathfrak{Cl}(\varphi)$ is continuous, then

$$\varphi = \iota_U^{-1} \circ \mathfrak{Cl}(\varphi) \circ \iota_V$$

   is continuous as a composition of continuous functions.

QED

Corollary 4.4.8. *The assignment $\mathfrak{C}\ell : \mathbf{CVS}_{\mathbb{K}} \to \mathbf{CVS}_{\mathbb{K}}$ is a functor and $\iota : id_{\mathbf{CVS}} \to \mathfrak{C}\ell$ is a natural transformation.*

From (3) of Theorem 4.4.7, we have the following:

Corollary 4.4.9. *If V is a convergence vector space and U is a locally convex topological convergence vector space, then as sets*

$$\mathcal{L}(V, W) = \mathcal{L}(\mathfrak{C}\ell(V), W)$$

and in particular

$$\mathcal{L}(V) = \mathcal{L}(\mathfrak{C}\ell(V)).$$

Proposition 4.4.10. *If V is a vector space and $\{V_i\}_{i\in I}$ is a family of locally convex topological convergence vector spaces and V is given the initial convergence structure relative to a family of linear maps $\{\varphi_i : V \to V_i\}$ then V is a locally convex topological CVS.*

Proof. Theorem 4.4.7 tells us that it is enough to show $\iota_V^{-1} : \mathfrak{C}\tau(V) \to V$ is continuous. By the universal property of the initial convergence structure this is continuous if and only if $\varphi_i \circ \iota_V^{-1} : \mathfrak{C}\tau(V) \to \mathfrak{C}(V_i)$ is continuous for each $i \in I$. The continuity of each $\varphi_i \circ \iota_V^{-1}$ follows from the commutativity of

$$\begin{array}{ccc} V & \xrightarrow{\quad \varphi_i \quad} & V_i \\ {\scriptstyle \iota_V^{-1}}\Big\uparrow & & \Big\uparrow{\scriptstyle \iota_{V_i}^{-1}} \\ \mathfrak{C}\ell(V) & \xrightarrow[\mathfrak{C}\ell(\varphi_i)]{} & \mathfrak{C}\ell(V_i) \end{array}$$

and the continuity of $\iota_{V_i}^{-1}$. We conclude that ι_V^{-1} is continuous and that V is a locally convex topological CVS. QED

Proposition 4.4.11. *Suppose V is a vector space equipped with the final vector space convergence structure relative to linear maps $\{\varphi_i : V_i \to V : i \in I\}$ from convergence vector spaces V_i. If U is a locally convex topological CVS, then a linear map $f : \mathfrak{C}\ell(V) \to U$ is continuous if and only if $f \circ \mathfrak{C}\ell(\varphi_i) : \mathfrak{C}\ell(V_i) \to U$ is continuous for each $i \in I$.*

Proof. If f is continuous, no work is required. Now, suppose $f \circ \mathfrak{C}\ell(\varphi_i) : \mathfrak{C}\ell(V_i) \to U$ is continuous for each $i \in I$. Define $f' : V \to U$ by $f' = f \circ i_V$. We then have the commutative diagram

$$\begin{array}{cccc} V_i & \xrightarrow{\quad \varphi_i \quad} & V & \\ {\scriptstyle \iota_{V_i}}\Big\downarrow & & \Big\downarrow{\scriptstyle \iota_V} & \searrow{\scriptstyle f'} \\ \mathfrak{C}\ell(V_i) & \xrightarrow[\mathfrak{C}\ell(\varphi_i)]{} & \mathfrak{C}\ell(V) & \\ & & & \searrow{\scriptstyle f} \\ & & & U \end{array}$$

for each $i \in I$. Observe that from the diagram we have

$$f' \circ \varphi_i = f \circ \mathfrak{C}\ell(\varphi_i) \circ \iota_{V_i}$$

which is continuous. Thus, by the universal property of the final vector space convergence structure (Proposition 4.2.13), we have that f' is continuous. Observe that $\mathfrak{C}\ell(f') = \iota_U \circ f$ is continuous since f' is. Since U is locally convex and topological, we have that ι_U^{-1} is continuous and so $f = \iota_U^{-1} \circ \mathfrak{C}\ell(f') = f$ is continuous as desired. QED

Definition 4.4.12. Suppose X is a convergence space and $f : X \to \mathbb{K}$ is some function. Define the *support of f* to be

$$\text{supp}(f) = \text{cl}(\{x \in X : f(x) \neq 0\}).$$

Theorem 4.4.13. *If X is a c-embedded space, then for each continuous seminorm $p : C_c(X) \to \mathbb{R}$ there is some $M > 0$ and compact $K \subseteq X$ so that*

1. *$p(f) \leq M \sup_K |f|$ for all $f \in C_c(X)$;*
2. *for each $x \in K$ and σ-open (that is, weakly open) $U \subseteq X$ with $x \in U$ there is $f \in C_c(X)$ so that supp$(f) \subseteq U$ and $p(f) \neq 0$.*

Proof. Fix a continuous seminorm $p : C_c(X) \to \mathbb{R}$.

First, suppose X is a compact topological space. Call a set $U \subseteq X$ *null* when for each $f \in C(X)$ with supp$(f) \subseteq U$ we have $p(f) = 0$. Define

$$N(p) = \bigcup \{U \subseteq X : U \text{ open and null}\}.$$

We claim that $N(p)$ is null. If there are no open null sets, then $N(p) = \emptyset$ which is certainly null. Suppose there are open null sets. Suppose $f \in C(X)$ with supp$(f) \subseteq N(p)$. Since supp(f) is compact, being a closed subset of a compact space, we have that supp$(f) \subseteq U_1 \cup U_2 \cup \cdots \cup U_n$ for some collection $U_1, U_2, ..., U_n$ of open null sets. Let $\{u_1, ..., u_n\}$ be a partition of unity subordinate to[6] $\{U_1, ..., U_n\}$. We then have that

$$\begin{aligned} p(f) &= p(u_1 f + u_2 f + \cdots + u_n f) \\ &\leq p(u_1 f) + p(u_2 f) + \cdots + p(u_n f) \\ &= 0 \end{aligned}$$

since each U_k is null and supp$(u_k) \subseteq U_k$. We thus conclude that $N(p)$ is null. Set $K = X \setminus N(p)$. If $K = \emptyset$, then (2) is vacuously satisfied. Else, suppose $x \in K$ and $U \subseteq X$ is a weakly open neighborhood of x. Since U is open and contains x, it cannot be null. Thus, there is some $f \in C_c(X)$ with supp$(f) \subseteq U$ and $p(f) \neq 0$. This is exactly property (2).

We now proceed to show property (1). First, we claim that

[6] See Appendix C.1 for details.

(a) there is some $c > 0$ so that $p(f) \leq c$ whenever $\sup|f| \leq 1$;

(b) if $f \in C(X)$ and $f(K) = \{0\}$ then $p(f) = 0$.

Consider the directed set

$$I = \{(f, \epsilon) : \epsilon \in \mathbb{R}_+ \text{ and } f \in C(X) \setminus \{0\} \text{ with } \sup|f| \leq \epsilon\}$$

ordered by $(f, \epsilon) \geq (g, \delta)$ when $\epsilon \leq \delta$. Let $\Lambda : I \to C_c(X)$ be a net with $\Lambda(f, \epsilon) = f$. Certainly we have $\Lambda \to 0$ in $C_c(X)$. Therefore, $p(\Lambda) \to 0$. Therefore, we may find $\epsilon > 0$ so that whenever $\sup|f| \leq \epsilon$ we have $p(f) \leq 1$. Thus, whenever $\sup|f| \leq 1$ we have $p(f) \leq \frac{1}{\epsilon}$. This is (a).

Now for (b), suppose $f \in C(X)$ and $f(K) = \{0\}$. For any $\epsilon > 0$ there is a function $g \in C(X)$ so that $g(x) = f(x)$ for all $x \in X$ with $|f(x)| \leq \epsilon$ and $|g(x)| = \epsilon$ otherwise. We next see that supp$(f - g) \subseteq X \setminus K$ (any converging net α on which $f - g$ is non-zero must converge to x with $|f(x)| \geq \epsilon$ and thus $x \notin K$). We have supp$(f - g) \subseteq N(p)$, so $p(f - g) = 0$. Lastly, have

$$\begin{aligned} p(f) &= p(f - g + g) \\ &\leq p(f - g) + p(g) \\ &< \epsilon c && \text{(by (a) since } \sup|g| \leq \epsilon) \end{aligned}$$

for all $\epsilon > 0$ so that $p(f) = 0$. This is (b).

Let $f \in C(X)$. If $\sup_K |f| = 0$, we have just shown that $p(f) = 0$. Else, we have that

$$\sup_K \left| \frac{f}{\sup_k |f|} \right| = 1$$

and so by (a) $p(f) \leq c \cdot \sup_k |f|$. This is (1).

Having shown the desired properties when X is a compact Hausdorff space, we now consider the case wherein X is a Tychonoff space. Write βX for the Stone-Čech compactification of X and $\eta_X : X \to \beta X$ for the continuous embedding of X into a dense subset of its compactification which exists since X is Tychonoff. Using this, we will think of X as a subspace of βX. Note that $\widehat{p} = p \circ \eta_X^* : C_c(\beta X) \to \mathbb{R}$ is a continuous seminorm.

By our first case, there is some $M > 0$ and compact $K \subseteq \beta X$ so that

- $\widehat{p}(f) \leq M \sup_K |f|$ for all $f \in C_c(\beta X)$;
- for each $x \in K$ and σ-open (that is, weakly open) $U \subseteq \beta X$ with $x \in U$ there is $f \in C_c(\beta X)$ so that supp$(f) \subseteq U$ and $\widehat{p}(f) \neq 0$.

We claim that $K \subseteq X$.

Suppose otherwise that there is some $x_0 \in K \setminus X$. Let $\mathcal{U}_0^\sigma$ denote the set of weakly open subsets of βX containing x_0, define $S_U = \{f \in C(\beta X) : f(\beta X \setminus U) = \{0\}\}$. Define the directed set

$$I = \{(f|_X, U) : f \in S_U, \quad U \in \mathcal{U}_0^\sigma\}$$

ordered by reverse inclusion on its second coordinate. Let $\Lambda : I \to C(X)$ be the net given by $\Lambda(f|_X, U) = f|_X$. Suppose $x \in X$ and there is a net α in X with $\alpha \to x$. Since βX is c-embedded, we have that it is functionally Hausdorff and since $x \neq x_0$, we may separate these points by weakly open sets. Thus, we have $\alpha \in_{\text{ev}} \beta X \setminus U$ for some weakly open $U \ni x_0$. Thus, $\Lambda(\alpha)$ is eventually 0. So, $\Lambda \to 0$ in $C(X)$ and $p(\Lambda) \to 0$ in $\mathbb{R}$.

Thus, we may find a weakly open set U containing x_0 so that $p(f|_X) \in [0, 1]$ whenever $f \in S_U$. Note that if $f \in S_U$ so are any of its scalar multiples. Thus, $p(f|_X) = 0$. By the first part, we know there is some $f \in C(\beta X)$ so that with $\operatorname{supp}(f) \subseteq U$ and $\widehat{p}(f) \neq 0$. Since $\operatorname{supp}(f) \subseteq U$, we have $f \in S_U$. Thus,

$$0 \neq \widehat{p}(f) = p \circ \eta_X^*(f) = p(f|_X) = 0.$$

This contradiction proves that $K \subseteq X$.

Now, suppose $x \in K$ and that $U \subseteq X$ is a weakly open neighborhood of x. We may then find an index set A and for each $\alpha \in A$ a natural number N_α, and for each α and $i = 1, ..., N_\alpha$ a continuous function $f_{\alpha,i} : X \to \mathbb{R}$ and an open set $V \subseteq \mathbb{R}$ so that

$$U = \bigcup_{\alpha \in A} \bigcap_{i=1}^{N_\alpha} f_{\alpha,i}^{-1}(V_{\alpha,i}).$$

For each α, i we may find index set $C_{\alpha,i}$ and for each $\gamma \in C_{\alpha,i}$ real numbers $r_{\alpha,i,\gamma}$ and $z_{\alpha,i,\gamma}$ so that

$$V_{\alpha,i} = \bigcup_{\gamma \in C_{\alpha,i}} B_{r_{\alpha,i,\gamma}}(z_{\alpha,i,\gamma}).$$

Putting all of this together,

$$U = \bigcup_{\alpha \in A} \bigcap_{i=1}^{N_\alpha} \bigcup_{\gamma \in C_{\alpha,i}} f_{\alpha,i}^{-1}(B_{r_{\alpha,i,\gamma}}(z_{\alpha,i,\gamma})).$$

Now, for each triple α, i, γ we may find a bounded continuous $f^*_{\alpha,i,\gamma} : X \to \mathbb{R}$ so that when $x \in X$ is such that $f_{\alpha,i}(x) \in B_{r_{\alpha,i,\gamma}}(z_{\alpha,i,\gamma})$ we have $f^*_{\alpha,i\gamma}(x) = f_{\alpha,i}(x)$ and so that when $f_{\alpha,i}(x) \notin B_{r_{\alpha,i,\gamma}}(z_{\alpha,i\gamma})$ we have $f^*_{\alpha,i,\gamma}(x) \notin B_{r_{\alpha,i,\gamma}}(z_{\alpha,i,\gamma})$. Thus,

$$U = \bigcup_{\alpha \in A} \bigcap_{i=1}^{N_\alpha} \bigcup_{\gamma \in C_{\alpha,i}} f^{*-1}_{\alpha,i,\gamma}(B_{r_{\alpha,i,\gamma}}(z_{\alpha,i,\gamma})).$$

For each $f^*_{\alpha,i,\gamma}$, we may find continuous $\widehat{f^*}_{\alpha,i,\gamma} : \beta X \to \mathbb{R}$ so that

$$\begin{array}{ccc} X & \xrightarrow{f^*_{\alpha,i,\gamma}} & \mathbb{R} \\ & {\scriptstyle \eta_X} \searrow \quad \nearrow {\scriptstyle \widehat{f^*}_{\alpha,i,\gamma}} & \\ & \beta X & \end{array}$$

commutes. Define weakly open $\widehat{U} \subseteq \beta X$ by

$$\widehat{U} = \bigcup_{\alpha \in A} \bigcap_{i=1}^{N_\alpha} \bigcup_{\gamma \in C_{\alpha,i}} \widehat{f^*}^{-1}_{\alpha,i,\gamma}(B_{r_{\alpha,i,\gamma}}(z_{\alpha,i,\gamma})).$$

Observe that $U = X \cap \widehat{U}$. There is some $f \in C(\beta X)$ so that $\mathrm{supp}(f) \subseteq \widehat{U}$ and $\widehat{p}(f) \neq 0$. We then have $f|_X \in C(X)$ and $\mathrm{supp}(f|_X) \subseteq U$ and $p(f|_X) \neq 0$. This is property (2) in the case of X Tychnoff.

We now prove property (1). Suppose $f \in C(X)$ is bounded. Note that since f is bounded, it has image contained in a compact set. We factor it through βX obtaining the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{R} \\ & {\scriptstyle \eta_X} \searrow \quad \nearrow {\scriptstyle \widehat{f}} & \\ & \beta X & \end{array}$$

We then have

$$p(f) = \widehat{p}(\widehat{f}) \leq M \sup_K |\widehat{f}| = M \sup_K |f|$$

since $K \subseteq X$. For f unbounded, we approximate f by a net Λ of bounded functions. The above inequality holds as p is continuous and $\sup_K |\Lambda|$ is eventually constant at value $\sup_K |f|$ due to compactness of K. This is property (1). This completes the proof in the case that X is Tychonoff.

We finally consider the full generality of X a c-embedded convergence space. We apply the last case to $\mathrm{tych}(X)$ - the Tychonoff modification of X. Recall that the quotient map $q : X \to \mathrm{tych}(X)$ induces a continuous mapping $q^{**} : CC(X) \to CC(\mathrm{tych}(X))$. Observe, $q^{**}(p) : C(\mathrm{tych}(X)) \to \mathbb{R}$ by $q^{**}(p)(f) = p(f \circ q)$ so that $q^{**}(p)$ is a continuous seminorm. Thus, by the previous case, there is some $M > 0$ and compact $K \subseteq \mathrm{tych}(X)$ so that

- $p(f \circ q) \leq M \sup_K |f|$ for all $f \in C_c(\mathrm{tych}(X))$;
- for each $x \in K$ and σ-open (that is, weakly open) $U \subseteq \mathrm{tych}(X)$ with $x \in U$ there is $f \in C_c(\mathrm{tych}(X))$ so that $\mathrm{supp}(f) \subseteq U$ and $p(f \circ q) \neq 0$.

We observe that each member of $C(X)$ is of the form $f \circ q$ for some $f \in C(\mathrm{tych}(X))$. Further, we observe that

$$\sup_K |f| = \sup_{q^{-1}(K)} |f \circ q|.$$

Further, if U is weakly open in $\mathrm{tych}(X)$, then $q^{-1}(U)$ is weakly open in X and $\mathrm{supp}(f \circ q) \subseteq q^{-1}(U)$. Thus it, only remains to show that $q^{-1}(K)$ is compact in X.

We will show that every covering system for $q^{-1}(K)$ in X has a finite subcover. Suppose $\mathcal{C}$ is a covering system for $q^{-1}(K)$. Suppose $\mathcal{F}$ is a filter with $\mathcal{F} \to x$ in X for some $x \in q^{-1}(K)$. Since X is functionally regular as it is c-embedded, we have that $\overline{\mathcal{F}}^{\sigma} \to x$. There is then some $A_{\mathcal{F}} \in \mathcal{F}$ and $C_{\mathcal{F}} \in \mathcal{C}$ so that $\overline{A_{\mathcal{F}}}^{\sigma} \subseteq C_{\mathcal{F}}$.

Now, note that since $\mathrm{tych}(X)$ is Hausdorff and K is compact, K is closed in $\mathrm{tych}(X)$. Thus, we have that $q^{-1}(K)$ is closed in X as well. Thus, for any filter $\mathcal{F}$ on X converging in $X \setminus q^{-1}(K)$, there is some $A_{\mathcal{F}} \in \mathcal{F}$ so that $\overline{A_{\mathcal{F}}}^{\sigma} \subseteq X \setminus q^{-1}(K)$.

Define the filter

$$\mathscr{F} = [\{T(A_{\mathcal{F}}, \{0\}) : \mathcal{F} \in \Phi(X) \text{ converges }\}]$$

recalling that

$$T(A_{\mathcal{F}}, \{0\}) = \{f \in C(X) : f(A_{\mathcal{F}}) \subseteq \{0\}\}.$$

We note that $\mathscr{F}$ is well defined since each element of its filter base contains at least the constant map at 0. Further, one has that $\mathscr{F} \to 0$ in $C(X)$ since for any converging $\mathcal{F} \in \Phi(X)$ we have that

$$\mathscr{F}(\mathcal{F}) \ni T(A_{\mathcal{F}}, \{0\})(A_{\mathcal{F}}) = \{0\}.$$

We then have that $p(\mathscr{F}) \to 0$ by continuity. There is then some $E \in \mathscr{F}$ so that $p(E) \subseteq [0,1]$. Without loss of generality, $E \in \mathscr{F}$ is a finite intersection of elements of the filter-base of $\mathscr{F}$ which means

$$E = T(A_{\mathcal{F}_1} \cup A_{\mathcal{F}_2} \cup \cdots \cup A_{\mathcal{F}_n}, \{0\})$$

for $\mathcal{F}_1, ..., \mathcal{F}_n$ filters converging in X. Observe that E is closed under scalar multiplication. It must then be that $p(E) = \{0\}$.

Suppose for the sake of contradiction that

$$x \in q^{-1}(K) \setminus A$$

where

$$A = \overline{A_{\mathcal{F}_1}}^{\sigma} \cup \overline{A_{\mathcal{F}_2}}^{\sigma} \cup \cdots \cup \overline{A_{\mathcal{F}_n}}^{\sigma}$$

which is weakly closed. Recall that any weakly closed set may be written as an arbitrary intersection of finite unions of preimages of closed sets. That is,

$$A = \bigcap_{\alpha} \bigcup_{i=1}^{N} f_{\alpha,i}^{-1}(T_{\alpha.i})$$

where each $f_{\alpha,i} : X \to \mathbb{K}$ continuous and $T_{\alpha,i}$ closed. Define

$$B = \bigcap_{\alpha} \bigcup_{i=1}^{N} \sigma(f_{\alpha,i})^{-1}(T_{\alpha.i})$$

which is weakly closed in $\mathrm{tych}(X)$. Observe that $q(x) \in B$ if and only if $x \in A$. Thus, $q(x) \in K \smallsetminus B$. There is then some $f \in C(X)$.

$$\text{supp}(\widetilde{f}) \subseteq \mathrm{tych}(X) \smallsetminus B$$

and $p(f) \neq 0$. But then $\widetilde{f} = \{0\}$ from which it follows that $f(A) = \{0\}$ and $f \in E$ so that $p(f) = 0$. This is a contradiction. Thus,

$$\{C_{\mathcal{F}_1}, C_{\mathcal{F}_2}, ..., C_{\mathcal{F}_n}\}$$

is a finite subset of $\mathcal{C}$ covering $q^{-1}(K)$. We conclude that $q^{-1}(K)$ is compact as desired to finish the proof. QED

4.5 Dual Spaces

In this section, we investigate the dual space of a convergence vector space.

Definition 4.5.1. If V is a convergence vector space, the *dual* of V is $\mathcal{L}(V)$, the space of continuous linear maps from V to the ground field $\mathbb{K}$. Unless otherwise specified, $\mathcal{L}(V)$ is assumed to carry the subspace convergence structure from $C_c(V, \mathbb{K})$. If we wish to emphasize that the dual carries the continuous convergence structure, we will write $\mathcal{L}_c(V)$ instead of $\mathcal{L}(V)$.

Recall that if X and Y are convergence spaces and $x \in X$, there is an evaluation at x map $\mathrm{ev}_x : C(X, Y) \to Y$ given by $f \mapsto f(x)$. Such evaluations induce another convergence structure on $\mathcal{L}(V)$.

Definition 4.5.2. If V is a convergence space, the *weak* convergence structure* on $\mathcal{L}(V)$ is the initial convergence structure relative to the family $\{\mathrm{ev}_v : \mathcal{L}(V) \to \mathbb{K} \mid v \in V\}$. We denote $\mathcal{L}V$ with the thereby induced convergence structure by $\mathcal{L}_\sigma(V)$.

Since $\mathbb{K}$ is topological, we have that $\mathcal{L}_\sigma(V)$ is topological. Its corresponding topology is the usual weak*-topology on $\mathcal{L}(V)$. Further, since each ev_v map is linear, $\mathcal{L}_\sigma(V)$ is a convergence vector space.

Notation 4.5.3. If $\mathscr{F}$ is a filter on $\mathcal{L}(V)$ and $v \in V$, define $\mathscr{F}(v) = \mathrm{ev}_v(\mathscr{F})$. If Λ is a net in $\mathcal{L}(V)$ and $v \in V$, define $\Lambda(v) = \mathrm{ev}_v(\Lambda)$.

Remark 4.5.4. Let $\mathscr{F}$ and Λ respectively be a filter and net in $\mathcal{L}(V)$ for some vector space V. If $v \in V$, then

1. $\mathcal{E}(\Lambda(v)) = \mathcal{E}(\Lambda)(v)$;
2. $\eta(\mathscr{F}(v)) \sim \eta(\mathscr{F})(v)$.

Proposition 4.5.5. *A net Λ on $\mathcal{L}_\sigma(V)$ converges to 0 if and only if for every $v \in V$ we have $\Lambda(v) \to 0$.*

Proof. If $\Lambda \to 0$, then $\Lambda(v) \to 0$ since $\mathrm{ev}_v : \mathcal{L}_\sigma(V) \to \mathbb{K}$ is continuous. Conversely, suppose $\Lambda(v) \to 0$ for each $v \in V$. Then $\Lambda \to 0$ since $\mathcal{L}_\sigma(V)$ is initial with respect to the evaluation maps ev_v. QED

Corollary 4.5.6. *A filter $\mathscr{F}$ on $\mathcal{L}_\sigma(V)$ converges to 0 if and only if for every $v \in V$ we have $\mathscr{F}(v) \to 0$.*

Corollary 4.5.7. *The "identity" map $\mathcal{L}_c(V) \to \mathcal{L}_\sigma(V)$ is continuous.*

Proof. Suppose we have a net $\Lambda \to 0$ in $\mathcal{L}_c(V)$. Let $v \in V$. Consider the net $\alpha : \{v\} \to V$ given by $\alpha(v) = v$. Certainly $\alpha \to v$. We have that $\Lambda(\alpha) \to 0$ in $\mathbb{K}$. We see that $\mathrm{ev}(\Lambda, \alpha) \sim \Lambda(v)$. Thus, $\Lambda(v) \to 0$ in $\mathbb{K}$. We conclude that $\Lambda \to 0$ in $\mathcal{L}_\sigma(V)$. Thus, the "identity" is continuous at 0 and therefore continuous. QED

This last result tells us that convergence in the continuous convergence structure is stronger than the pointwise convergence of the weak* structure.

With additional information, it is possible to use weak* convergence to obtain convergence in $\mathcal{L}_c(V)$. We will need to introduce a definition and prove an auxiliary lemma to show this.

Definition 4.5.8. If V is a vector space, the *polar* of a subset $U \subseteq V$ is

$$U^\circ = \{\varphi \in \mathcal{L}(V) : \forall u \in U \ \ |\varphi(u)| \leq 1\}.$$

Lemma 4.5.9. *If $\mathscr{F}$ is a filter on $\mathcal{L}(V)$ and $\mathcal{G}$ a filter on V and $v \in V$, then*

$$\mathrm{ev}(\mathscr{F} \times \mathcal{G}) \supseteq \mathrm{ev}(\mathscr{F} \times (\mathcal{G} - v)) + \mathscr{F}(v).$$

or in alternative notation

$$\mathscr{F}(\mathcal{G}) \supseteq \mathscr{F}(\mathcal{G} - v) + \mathscr{F}(v).$$

Proof. Suppose $H \in \mathscr{F}(\mathcal{G} - v) + \mathscr{F}(v)$. There are then $F_1, F_2 \in \mathscr{F}$ and $G \in \mathcal{G}$ so that $H \supseteq F_1(G - v) + F_2(v)$. We thus have

$$\begin{aligned} H &\supseteq F_1(G - v) + F_2(v) \\ &\supseteq (F_1 \cap F_2)(G - v) + (F_1 \cap F_2)(v) \\ &\supseteq (F_1 \cap F_2)(G - v + v) \\ &= (F_1 \cap F_2)(G) \\ &\in \mathscr{F}(\mathcal{G}) \end{aligned}$$

which implies that $H \in \mathscr{F}(\mathcal{G})$ as desired. QED

We can now give the conditions under which weak* convergence implies convergence in $\mathcal{L}_c(V)$.

Proposition 4.5.10. *Let V be a CVS. A filter $\mathscr{F}$ converges in $\mathcal{L}_c(V)$ if and only if it converges in $\mathcal{L}_\sigma(V)$ and for every filter $\mathcal{G} \to 0$ in V there is $G \in \mathcal{G}$ so that $G^\circ \in \mathscr{F}$.*

Proof. Since the "identity" $\mathcal{L}_c(V) \to \mathcal{L}_\sigma(V)$ is continuous, we have that any filter $\mathscr{F}$ converging in $\mathcal{L}_c(V)$ converges in $\mathcal{L}_\sigma(V)$. It remains to show the condition on polars. This may be rephrased as for every net $\alpha \to 0$ in V and any net $\Lambda \to \varphi$ in $\mathcal{L}_c(V)$ we have some U which eventually contains α and $\Lambda \in_{\text{ev}} U^\circ$. So let $\alpha : A \to V$ and $\Lambda : L \to \mathcal{L}(V)$ be such nets. Suppose for the sake of contradiction that the desired condition fails. That is, for each U in the eventuality filter of α we have $\Lambda \notin_{\text{ev}} U^\circ$. Thus, for every $a \in A$ we have that $\Lambda \notin_{\text{ev}} (T_a\alpha)^\circ$ where $T_a\alpha$ is the tail

$$T_a\alpha = \{\alpha_i : i \in A \quad \text{and} \quad \geq a\}.$$

Thus, for every $a \in A$ and $\ell_0 \in L$ there is $\ell \geq \ell_0$ so that $|\Lambda_\ell(\alpha_a)| > 1$. But then it is clearly the case that $\Lambda(\alpha) \notin_{\text{ev}} \mathbb{D}$. But then $\Lambda(\alpha) \not\to 0$ in $\mathbb{K}$ which contradicts the convergence of Λ in $\mathcal{L}_c(V)$. Thus, we have the desired condition on polars.

Conversely, suppose that we have a filter $\mathscr{F} \to \varphi$ in $\mathcal{L}_\sigma(V)$ and for every filter $\mathcal{G} \to 0$ in V there is $G \in \mathcal{G}$ so that $G^\circ \in \mathscr{F}$. We wish to show that $\mathscr{F} \to \varphi$ in $\mathcal{L}_c(V)$. It suffices to show that for all $v \in V$ and filters $\mathcal{G} \to v$ we have $\mathscr{F}(\mathcal{G}) \to \varphi(v)$. Fix such a $v \in V$ and filter $\mathcal{G}$ on V. Fix $\epsilon > 0$. We have that $\frac{1}{\epsilon}(\mathcal{G} - v) \to 0$. We then have some $U \in \frac{1}{\epsilon}(\mathcal{G} - v)$ so that $U^\circ \in \mathscr{F}$. We then have $\epsilon U \in (\mathcal{G} - v)$ and $\text{ev}(U^\circ \times \epsilon U) \in \mathscr{F}(\mathcal{G} - v)$. Further, $\text{ev}(U^\circ \times \epsilon U) \subseteq \epsilon\mathbb{D}$. Therefore, $\epsilon\mathbb{D} \in \mathscr{F}(\mathcal{G} - v)$. As this holds for each $\epsilon > 0$ we have $\mathscr{F}(\mathcal{G} - v) \to 0$ in $\mathbb{K}$. We conclude that since

$$\mathscr{F}(\mathcal{G}) \supseteq \mathscr{F}(\mathcal{G} - v) + \mathscr{F}(v).$$

by Lemma 4.5.9 and $\mathscr{F}(v) \to \varphi(v)$ since $\mathscr{F} \to \varphi$ in $\mathcal{L}_\sigma(V)$ it must be the case that $\mathscr{F}(\mathcal{G}) \to \varphi(v)$ as desired. QED

Corollary 4.5.11. *A filter $\mathscr{F} \to 0$ in $\mathcal{L}_c(V)$ if and only if it contains the polar of each finite subset of V and for every filter $\mathcal{G} \to 0$ in V there is $G \in \mathcal{G}$ so that $G^\circ \in \mathscr{F}$.*

Proof. We need only check that containment of the polar of a finite set is equivalent to convergence to 0 in $\mathcal{L}_\sigma(V)$. Thus, let Λ be a net in $\mathcal{L}_c(V)$ which converges to 0. Pick $F \subseteq V$ finite. We have that $\Lambda(v) \to 0$ for each $v \in F$. We thus have that $\Lambda(v) \in_{\text{ev}} \mathbb{D}$ for each $v \in F$. It is then clear that $\Lambda \in_{\text{ev}} F^\circ$ by taking the maximum required index.

Conversely, assume that $\Lambda \in_{\text{ev}} F^\circ$ for each finite $F \subseteq V$. Particularly, given any $v \in V$ and $\epsilon > 0$ we have $\Lambda \in_{\text{ev}} \{(1/\epsilon)v\}^0$. Thus, $\Lambda(v) \in_{\text{ev}} \epsilon\mathbb{D}$ and we conclude that $\Lambda(v) \to 0$ for each $v \in V$. This means that $\Lambda \to 0$ in $\mathcal{L}_\sigma(V)$. QED

Definition 4.5.12. If V is a normed space, $\mathcal{L}(V)$ can be made into a normed space with the *operator norm*

$$||\varphi|| = \sup\{|\varphi(v)| : ||v|| \leq 1\}$$

for all $v \in V$. We denote $\mathcal{L}(V)$ equipped with the corresponding convergence vector space structure as $\mathcal{L}_{\text{op}}(V)$ and denote convergence in this setting by $\to_{\text{op}}$.

Remark 4.5.13. If V is a normed space and $\mathscr{F}$ and Λ are a filter and net respectively on $\mathcal{L}(V)$, then

1. $\mathscr{F} \to_{\text{op}} 0$ if and only if $||\mathscr{F}|| \to 0$ in $\mathbb{R}$;

2. $\Lambda \to_{\text{op}} 0$ if and only if $||\Lambda|| \to 0$ in $\mathbb{R}$.

Remark 4.5.14. If V is a normed space, it is well known that the identity $\mathcal{L}_{\text{op}}(V) \to \mathcal{L}_{\sigma}(V)$ is continuous.

Proposition 4.5.15. *If V is a normed space, the identity map $\mathcal{L}_{\text{op}}(V) \to \mathcal{L}_c(V)$ is continuous.*

Proof. Since the map in question is a linear map between convergence vector spaces, it suffices to show that it is continuous at 0. Fix a filter $\mathscr{F}$ on $\mathcal{L}(V)$ so that $\mathscr{F} \to_{\text{op}} 0$. Suppose $\mathcal{G}$ is a filter on V with $\mathcal{G} \to 0$. From this it follows that $\mathbb{D} \in ||\mathcal{G}||$ and that there is some $G \in \mathcal{G}$ so that $||G|| \subseteq \mathbb{D}$. Fix $\epsilon > 0$. Since $\mathscr{F} \to_{\text{op}} 0$, we have that $\epsilon\mathbb{D} \in ||\mathscr{F}||$. There is thus $F \in \mathscr{F}$ so that $||F|| \subseteq \epsilon\mathbb{D}$. Thus, we have $F(G) \subseteq \epsilon\mathbb{D}$ and $F(G) \in \mathscr{F}(\mathcal{G})$ so that $\epsilon\mathbb{D} \in \mathscr{F}(\mathcal{G})$. As this holds for all $\epsilon > 0$, we have established $\mathscr{F}(\mathcal{G}) \to 0$.

Now, if $\mathcal{H}$ is a filter on V and $\mathcal{H} \to v$ for some $v \in V$, we have that $\mathcal{H} - v \to 0$. By Lemma 4.5.9, there holds

$$\mathscr{F}(\mathcal{H}) \supseteq \mathscr{F}(\mathcal{H} - v) + \mathscr{F}(v).$$

By the first part of this proof, $\mathscr{F}(\mathcal{H}-v) \to 0$ and by Remark 4.5.14 $\mathscr{F}(v) \to 0$. Thus, $\mathscr{F}(\mathcal{H}) \to 0$. As this holds for all converging filters $\mathcal{H}$ on V, we have that $\mathscr{F} \to 0$. This establishes the desired continuity. QED

We recall that if $f : X \to Y$ is a continuous mapping of convergence spaces, then there is a continuous map $f^* : C(Y) \to C(X)$. A similar construction is available for dual spaces.

Definition 4.5.16. Let V, W be CVSs and $f : V \to W$ be a continuous linear map. Then map

$$f' : \mathcal{L}(W) \to \mathcal{L}(V)$$

given by $f'(\varphi) = \varphi \circ f$ is called the *adjoint mapping* of f.

Proposition 4.5.17. *Let V, W be CVSs and $f : V \to W$ be a continuous linear map.*

1. *Then $f' : \mathcal{L}_c(W) \to \mathcal{L}_c(V)$ is linear and continuous.*

2. $\ker(f') = f(V)^{\perp}$.

3. *f' is an injection if and only if $f(V)$ is dense in $\mathfrak{Cl}(W)$.*

4. *$(g \circ f)' = f' \circ g'$ and $(id_V)' = id_{\mathcal{L}(V)}$.*

Proof. (1) To show f' is continuous, it suffices by Corollary 3.2.11 to show that its transpose $T_1(f')$ continuous. Continuity of the transpose follows from commutativity of

$$\begin{array}{ccc} \mathcal{L}_c(W) \times V & & \\ {\scriptstyle \mathrm{id} \times f}\downarrow & \searrow^{T_1(f')} & \\ \mathcal{L}_c(W) \times W & \xrightarrow[\mathrm{ev}]{} & \mathbb{K} \end{array}$$

To see that f' is linear, let $\varphi, \psi \in \mathcal{L}(W)$ and $\alpha, \beta \in \mathbb{K}$. We see that

$$\begin{aligned} f'(\alpha\varphi + \beta\psi) &= (\alpha\varphi + \beta\psi) \circ f \\ &= \alpha(\varphi \circ f) + \beta(\psi \circ f) \\ &= \alpha f'(\varphi) + \beta f'(\psi). \end{aligned}$$

(2) Suppose $\varphi \in \ker(f')$. Let $v \in V$. We see that $\varphi(f(v)) = f'(\varphi)(v) = 0$ so that $\varphi \in f(V)^\perp$. Now, suppose that $\varphi \in f(V)^\perp$. We see that $f'(\varphi)(v) = 0$ for all $v \in V$, so $\varphi \in \ker(f')$. Thus, $\ker(f') = f(V)^\perp$ as desired.

(3) Suppose $f(V)$ is dense in $\mathfrak{Cl}(W)$. Suppose $\varphi \in \mathcal{L}(W)$ so that $f'(\varphi) = 0$. Let $w \in W$. Since $f(V)$ is dense in $\mathfrak{Cl}(W)$, we have some net α in V so that $f(\alpha) \to w$ in $\ell(W)$. Recall that by Corollary 4.4.9 we know $\mathfrak{Cl}(W)$ and W have the same linear functionals, so φ is continuous as a map out of $\mathfrak{Cl}(W)$. Therefore, $\varphi \circ f(\alpha) \to \varphi(w)$. Since $f'(\varphi) = 0$, we have that the constant 0 net in $\mathbb{K}$ converges to $\varphi(w)$. Thus, $\varphi(w) = 0$. It is then the case that $\varphi = 0$ and we have that f' is an injection.

Now, for the converse, suppose that f' is an injection. We then have by (2) that $f(V)^\perp = \{0\}$ and thus that $(f(V)^\perp)^\perp = W$. But we also have by Proposition D.4.14 that $(f(V)^\perp)^\perp = \overline{f(V)}$ where the closure is taken in $\ell(V)$. The result follows.

(4) is a straightforward computation. QED

Remark 4.5.18. It follows from (1) and (4) that the assignments $V \mapsto \mathcal{L}_c(V)$ and $f \mapsto f'$ give a contravariant functor from the category of convergence $\mathbb{K}$-vector space to itself.

We will next see how the dual space interacts with various vector space constructions, *e.g.* the final vector space convergence structure, products, and coproducts.

Theorem 4.5.19. *If V is a CVS carrying the final CVS structure relative to a family $\{\varphi_i : V_i \to V\}_{i \in I}$ of linear maps so that V is the span of $\bigcup_{i \in I} \varphi_i(V_i)$, then $\mathcal{L}_c(V)$ carries the initial convergence structure relative to the family $\{\varphi_i' : \mathcal{L}(V) \mapsto \mathcal{L}(V_i)\}_{i \in I}$.*

Proof. Fix $f \in \mathcal{L}_c(V)$. We must show that a filter $\mathscr{F}$ converges to f if and only if $\varphi_i'(\mathscr{F}) \to \varphi_i'(f)$ for all $i \in I$. Since each φ_i is continuous and linear, we have that each φ_i' is as well by Proposition 4.5.17. Thus, if a filter $\mathscr{F}$ converges to f in $\mathcal{L}_c(V)$, then it certainly must be that $\varphi_i'(\mathscr{F}) \to \varphi_i'(f)$ for all $i \in I$.

We now consider the converse. Suppose $\varphi_i'(\mathscr{F}) \to \varphi_i'(f)$ for all $i \in I$. To show that $\mathscr{F} - f \to 0$, we will show that $\mathscr{F} - f$ contains the polar of each finite subset of V and for every filter $\mathcal{G} \to 0$ in V there is $G \in \mathcal{G}$ so that $G^\circ \in \mathscr{F} - f$. This suffices due to Corollary 4.5.11.

Let $N = \{x^{(1)}, x^{(2)}, ..., x^{(n)}\} \subseteq V$. For each $i \in I$ and $k = 1, ..., n$, we may find $v_i^{(k)} \in V_i$ only finitely many of which are non-zero so that $x^{(k)} = \sum_{i \in I} \varphi_i(v_i^{(k)})$ since V is the span of $\bigcup_{i \in I} \varphi_i(V_i)$. Let $M \in \mathbb{Z}^+$ be larger than the number of non-zero terms appearing in these sums. Fixing $i \in I$, since $\varphi_i'(\mathscr{F} - f) \to 0$, we have that $\varphi_i'(\mathscr{F} - f)$ contains the polar of $N_i = \{Mv_i^{(1)}, Mv_i^{(2)}, ..., Mv_i^{(n)}\}$ by Corollary 4.5.11. There is thus some $F_i \in \mathscr{F} - f$ so that $\varphi_i'(F_i) \subseteq N_i^\circ$. Observe that for all but finitely many $i \in I$, we have $N_i = \{0\}$ so that $N_i^\circ = \mathcal{L}(V_i)$ in the case that $N_i = \{0\}$. In this case, we choose $F_i = \mathcal{L}(V)$. Thus, $F = \bigcap_{i \in I} F_i$ is (equal to) a finite intersection of elements of $\mathscr{F} - f$ and we may conclude $F \in \mathscr{F} - f$. We now claim that $F \subseteq N^\circ$. Let $k = 1, ..., n$ and $h \in F$. We have that

$$\begin{aligned} h(x^{(k)}) &= \sum_{i \in I} h \circ \varphi_i(v_i^{(k)}) \\ &= \sum_{i \in I} \varphi_i'(h)(v_i^{(k)}). \end{aligned}$$

We thus have that

$$\begin{aligned} |h(x^{(k)})| &\leq \sum_{i \in I} |\varphi_i'(h)(v_i^{(k)})| \\ &= \sum_{i \in I} (1/M)|\varphi_i'(h)(Mv_i^{(k)})| \\ &< \sum_{i \in I^*} 1/M \qquad \text{(with } I^* \text{ indexing the non-zero summands)} \\ &\leq M(1/M) \\ &= 1. \end{aligned}$$

Therefore, $F \subseteq N^\circ$ and $N^\circ \in \mathscr{F} - f$.

Now, let $\mathcal{G} \to 0$ in V. By Corollary 4.2.19 we may find finitely many indices $J \subseteq I$ and for each $j \in J$ a filter $\mathcal{G}_j$ converging to 0 in V_j so that

$$\mathcal{G} \supseteq \sum_{j \in J} \varphi_j(\mathcal{G}_j).$$

For each $j \in J$, we have that $\varphi_j'(\mathscr{F} - f) \to 0$. Thus, by Corollary 4.5.11, since $|J|\mathcal{G}_j \to 0$, there is $G_j \in \mathcal{G}_j$ so that $(|J|G_j)^\circ \in \varphi_j'(\mathscr{F} - f)$. There is thus some $F_j \in \mathscr{F} - f$ so that $\varphi_j'(F_j) \subseteq (|J|G_j)^\circ$. Let $F = \bigcap_{j \in J} F_j$ and let $G = \sum_{j \in J} \varphi_j(G_j) \in \mathcal{G}$. Let $h \in F$ and $x \in G$. For each $j \in J$, we may find $v_j \in G_j$ so that $x = \sum_{j \in J} \varphi_j(v_j)$. We then have

$$\begin{aligned}|h(x)| &\leq \sum_{j \in J} |h \circ \varphi_j(v_j)| \\ &= \sum_{j \in J} (1/|J|)|h \circ \varphi_j(|J|v_j)| \\ &= \sum_{j \in J} (1/|J|)|\varphi_j'(h)(|J|v_j)| \\ &< 1.\end{aligned}$$

Therefore, $h \in G^\circ$ so that $F \subseteq G^\circ$. Therefore, $G^\circ \in \mathscr{F} - f$.

We now may conclude that $\mathscr{F} - f \to 0$ in $\mathcal{L}_c(V)$ as desired. QED

Corollary 4.5.20. *If V and W are CVSs $\pi : V \to W$ is a quotient map, then $\pi' : \mathcal{L}_c(W) \to \mathcal{L}_c(V)$ is an embedding onto $(\ker \pi)^\perp$.*

Proof. We first show that π' is an injection and that $\pi'^{-1} : \pi'(\mathcal{L}_c(W)) \to \mathcal{L}_c(W)$ is continuous.

Since π is surjective, we have by Proposition 4.5.17 that

$$\ker \pi' = \pi(V)^\perp = W^\perp = \{0\}$$

so that π' is an injection.

By Theorem 4.5.19, we have that $\mathcal{L}_c(W)$ carries the initial convergence with respect to π'. We then have that $\pi'^{-1} : \pi'(\mathcal{L}_c(W)) \to \mathcal{L}_c(W)$ is continuous by the universal property of initial convergence applied to the fact that $\pi' \circ \pi'^{-1}$ is merely the injection of $\pi'(\mathcal{L}_c(W))$ into $\mathcal{L}_c(V)$ which is continuous.

Now, suppose that $\varphi \in \mathcal{L}_c(W)$. Suppose that $v \in \ker \pi$. We then have that $\pi'(\varphi)(v) = \varphi \circ \pi(v) = 0$ so that $\pi'(\varphi) \in (\ker \pi)^\perp$. Now, suppose that $\psi \in \mathcal{L}_c(V)$ and $\psi \in (\ker \pi)^\perp$. By the universal property of quotients, there is some $\psi' \in \mathcal{L}_c(W)$ so that $\psi = \psi' \circ \pi = \pi'(\psi')$. Thus, $\pi'(\mathcal{L}(W)) = (\ker \pi)^\perp$. QED

Corollary 4.5.21. *If V is a CVS with subspace M, then $\mathcal{L}_c(V/M) \cong M^\perp$.*

Theorem 4.5.22. *If $\{V_i\}_{i \in I}$ is a family of CVSs, then there is an isomorphism*

$$u : \mathcal{L}_c\left(\prod_{i \in I} V_i\right) \to \bigoplus_{i \in I} \mathcal{L}_c(V_i)$$

where for each linear functional f in the domain we have $u(f) = (f \circ e_i)_{i \in I}$ and $e_i : V_i \to \prod_{i \in I} V_i$ is the usual injection.

Proof. We first must address well definition. Certainly, for any $i \in I$ and continuous linear $f : \prod_{i \in I} V_i \to \mathbb{K}$, the coordinate maps $f \circ e_i : V_i \to \mathbb{K}$ are continuous linear functionals. We must check that only finitely many are non-zero. Suppose not, that is for some infinite $J \subseteq I$ and each $j \in J$ there is $v_j \in V_j$ so that $f(e_j(v_j)) \neq 0$. Without loss of generality, we may have $f(e_j(v_j)) = 1$ and $J = \{j_0, j_1, ...\}$ countable. For each $n \in \mathbb{N}$, define $z_n \in \prod_{i \in I} V_i$ by for each $i \in I$

$$(z_n)_i = \begin{cases} 0 & i \notin J \text{ or } i = j_k \in J \text{ and } k < n \\ v_i & \text{else} \end{cases}$$

By this definition, we have that for all $n \in \mathbb{N}$

$$z_0 = z_n + \sum_{k<n} e_j(v_{j_k})$$

and

$$f(z_0) = f(z_n) + n.$$

Consider the net $\alpha : \mathbb{N} \to \prod_{i \in I} V_i$ given by $\alpha_n = z_n$. We have that $\alpha \to 0$ since each coordinate is eventually zero. By continuity of f, we then have that $f(\alpha) \to 0$. But this is impossible since $f(z_n) = f(z_0) - n$. From this contradiction, we have that $f \circ e_i$ must be zero for all but finitely many $i \in I$. We then have that u is well defined.

We will next show that u is an injection. Suppose that f is a continuous linear functional on $\prod_{i \in I} V_i$ with $f \in \ker u$. Place a well ordering $\leq$ on I with minimum s and maximum t. Let $v \in \prod_{i \in I} V_i$. Define a net $\alpha : I \to \prod_{i \in I} V_i$ so that the i-th coordinate of α_j is given by

$$(\alpha_j)_i = \begin{cases} 0 & i < j \\ v_i & j \leq i \end{cases}$$

for all $i, j \in I$. Observe that $\alpha \to \alpha_t$. Thus by continuity of $f \in \ker u$, we have $f(\alpha) \to 0 = f(\alpha_t)$. We claim that $f(\alpha)$ is constant with value $f(\alpha_s)$. Suppose to contradiction that $f(\alpha)$ is not constant. Since I is well ordered, there is least $s < i_0 \in I$ so that $f(\alpha_{i_0}) \neq f(\alpha_s)$. We distinguish two cases, i_0 is the successor of some i_0' or not. First, suppose i_0 is the successor of i_0'. Let $x \in \prod_{i \in I} V_i$ be given by

$$x_i = \begin{cases} (\alpha_{i_0'})_i & i = i_0' \\ 0 & \text{else} \end{cases}$$

for each $i \in I$. We have that $\alpha_{i_0} = \alpha_{i_0'} - x$. We see that $f(x) = 0$. Thus, $f(\alpha_{i_0}) = f(\alpha_{i_0'}) = f(\alpha_s)$. Now, suppose that i_0 has no predecessor. Let $J = \{j \in I : j < i_0\}$. We have that $\alpha|_J \to \alpha_{i_0}$. Then we have $f(\alpha|_J) \to f(\alpha_{i_0})$. But since this is a constant net in $\mathbb{K}$ with value $f(\alpha_s)$, we have $f(\alpha_{i_0}) = f(\alpha_s)$. This contradiction proves that

$f(\alpha)$ is constant. We now have that $f(\alpha)$ is a constant sequence converging to 0. Therefore, $f(\alpha_s) = 0$. Since $\alpha_s = v$, we have that $f(v) = 0$. This holds for all $v \in V$, so we have $f = 0$. Therefore, $\ker u = \{0\}$ and u is an injection.

We now show that u is a surjection. Let $f = (f_i)_{i \in I} \in \bigoplus_{i \in I} \mathcal{L}_c(V_i)$. Note that only finitely many of the $f_i : V_i \to \mathbb{K}$ may be non-zero. Let $g : \prod_{i \in I} V_i \to \mathbb{K}$ by

$$g = \sum_{i \in I} f_i \circ \pi_i$$

where $\pi_i : \prod_{i \in I} V_i \to V$ are the usual projection maps. This is well defined since only finitely many f_i are non-zero. Further, this is continuous and linear since it is a sum of compositions of such functions. It is clear that $u(g) = f$

We lastly show that u is continuous. Since u is linear, it suffices to show that u is continuous at 0. Consider a filter $\mathscr{F} \to 0$ in $\mathcal{L}_c\left(\prod_{i \in I} V_i\right)$. Define a filter on $\prod_{i \in I} V_i$ given by

$$\mathcal{G} = \left\{ G \subseteq \prod_{i \in I} V_i \;\middle|\; \begin{array}{l} G \supseteq \prod_{i \in I} A_i \text{ so that } A_i = V_i \text{ for all but finitely many } i \in I, \\ \text{and } A_i = \{0\} \text{ otherwise} \end{array} \right\}$$

We have that $\mathcal{G} \to 0$ since $\pi_i(\mathcal{G}) = [0]$ for all $i \in I$. Thus, $\mathscr{F}(\mathcal{G}) \to 0$ in $\mathbb{K}$, and there is some $F_0 \in \mathscr{F}$ and $G_0 \in \mathcal{G}$ so that $F_0(G_0) \subseteq \mathbb{D}$. By construction of $\mathcal{G}$, we may find finite $J \subseteq I$ so that when

$$A_i = \begin{cases} \{0\} & i \in J \\ V_i & i \in I \setminus J \end{cases}$$

we have

$$F_0\left(\prod_{i \in I} A_i\right) \subseteq \mathbb{D}.$$

In particular, for each $i \in I \setminus J$ and $n \in \mathbb{N}$ and $v \in V_i$, we have that $F_0(e_i(nv)) \subseteq \mathbb{D}$. Since $F_0(e_i(nv)) = nF_0(e_i(v))$, it must be that $F_0(e_i(V_i)) = 0$ for all $i \in I \setminus J$.

For each $i \in I$, we know that the adjoint $e_i' : \mathcal{L}_c\left(\prod_{i \in I} V_i\right) \to \mathcal{L}_c(V_i)$ is linear and continuous. We thus have that $e_i'(\mathscr{F}) \to 0$ in $\mathcal{L}_c(V_i)$ for all $i \in I$. For each $i \in I$, let $E_i : \mathcal{L}_c(V_i) \to \bigoplus_{i \in I} \mathcal{L}_c(V_i)$ be the usual injection. As this is linear and continuous, $E_i(e_i'(\mathscr{F})) \to 0$. We now see that

$$u(\mathscr{F}) \supseteq \sum_{j \in J} E_j(e_j'(\mathscr{F}))$$

since for any $F \in \mathscr{F}$,

$$\begin{aligned}
\sum_{j\in J} E_j(e_j'(F)) &= \Big\{ \sum_{j\in J} E_j(e_j'(f_j)) : f_j \in F \Big\} \\
&\supseteq \Big\{ \sum_{j\in J} E_j(e_j'(f)) : f \in F \Big\} \\
&= \Big\{ \sum_{j\in J} E_j(f \circ e_j) : f \in F \Big\} \\
&\supseteq \Big\{ \sum_{j\in J} E_j(f \circ e_j) : f \in F \cap F_0 \Big\} \\
&= \{ (f \circ e_i)_{i\in I} : f \in F \cap F_0 \} \\
&= u(F \cap F_0) \\
&\in u(\mathscr{F}).
\end{aligned}$$

Therefore, $u(\mathscr{F}) \to 0$ in $\bigoplus_{i\in I} \mathcal{L}_c(V_i)$ and u is continuous.

We must lastly show that u^{-1} is continuous. One can see for $f = (f_i)_{i\in I} \in \bigoplus_{i\in I} \mathcal{L}_c(V_i)$ that

$$u^{-1}(f) = \sum_{i\in I} f_i \circ \pi_i.$$

To show that u^{-1} is continuous, it suffices to show that for all $i \in I$, the composition $u^{-1} \circ E_i : \mathcal{L}_c(V_i) \to \mathcal{L}_c\left(\prod_{i\in I} V_i\right)$ is continuous. Note that if $i, j \in I$ and $f \in \mathcal{L}_c(V_i)$, then

$$(E_i f)_j \circ \pi_j = \begin{cases} f \circ \pi_i & i = j \\ 0 & i \neq j \end{cases}$$

Thus, $u^{-1} \circ E_i = \pi_i'$ which is certainly continuous.

We have now that u is a linear homeomorphism as desired. QED

Theorem 4.5.23. *If $\{V_i\}_{i\in I}$ is a family of convergence vector spaces, then*

$$u : \mathcal{L}_c\left(\bigoplus_{i\in I} V_i\right) \to \prod_{i\in I} \mathcal{L}_c(V_i)$$

given by $u(f) = (f \circ e_i)_{i\in I}$, where $e_j : V_j \to \bigoplus_{i\in I} V_i$ is the defining injection, is a linear homeomorphism.

Proof. It is clear that u is linear and well defined.

We will show that u is a bijection. Suppose $f \in \ker u$. We then have that $f \circ e_i = 0$

for all $i \in I$, but since $\bigoplus_{i\in I} V_i$ is spanned by $\bigcup_{i\in I} e_i(V_i)$, we have that $f = 0$. Therefore, u is an injection. Now, suppose that $(f_i)_{i\in I} \in \prod_{i\in I} \mathcal{L}_c(V_i)$. For each $j \in I$, let $\pi_j : \bigoplus_{i\in I} V_i \to V_j$ be the usual projection. Define $g : \bigoplus_{i\in I} V_i \to \mathbb{K}$ by

$$g(v) = \sum_{i\in I} f_i \circ \pi_i(v)$$

which is linear. To show that g is continuous, it suffices to show that each composition $g\circ e_i$ is continuous. This composition is continuous because $g\circ e_i = f_i \in \mathcal{L}_c(V_i)$. Further, it is clear that $u(g) = f$ so that u is surjective.

We will next show that u is continuous. As u is a map into a product space, we need merely show that $P_j \circ u$ is continuous for each projection $P_j : \prod_{i\in I} \mathcal{L}_c(V_i) \to \mathcal{L}_c(V_j)$. Notice that $P_j \circ u = e_j'$ which is continuous. Note that

$$\begin{array}{ccc}
\mathcal{L}_c\left(\bigoplus_{i\in I} V_i\right) & \xrightarrow{\;e_j'\;} & \mathcal{L}_c(V_j) \\
{\scriptstyle u^{-1}}\big\uparrow & \nearrow_{P_j} & \\
\prod_{i\in I} \mathcal{L}_c(V_i) & &
\end{array}$$

commutes for all $j \in I$.

We finally show that u^{-1} is continuous. Suppose that $\mathscr{F} \to 0$ in $\prod_{i\in I} \mathcal{L}_c(V_i)$. For all $j \in J$ we must then have $P_j(\mathscr{F}) \to 0$. Therefore, $e_j' \circ u^{-1}(\mathscr{F}) \to 0$. Since $\mathcal{L}_c\left(\bigoplus_{i\in I} V_i\right)$ is initial with respect to the e_j' by Theorem 4.5.19, we have that $u^{-1}(\mathscr{F}) \to 0$. Therefore, u^{-1} is continuous at 0 and thus everywhere.

We conclude that u is a linear homeomorphism. QED

The following proposition is rightly a corollary to Theorem 4.4.13 which was delayed as it requires adjoint mappings.

Proposition 4.5.24. *If X is a c-embedded space and for each $K \subseteq X$ compact we denote by $e_K : K \to X$ the inclusion map, then*

$$\mathcal{L}\,C_c(X) = \bigcup \{(e_K^*)'(\mathcal{L}\,C_c(K))) : K \subseteq X \text{ compact }\}.$$

Proof. Clearly, it suffices to show that

$$\mathcal{L}\,C_c(X) \subseteq \bigcup \{(e^*)'(\mathcal{L}\,C_c(K))) : K \subseteq X \textit{ compact }\}.$$

Fix $\varphi \in \mathcal{L}\,C(X)$. Then $|\varphi| : C(X) \to \mathbb{R}$ is a continuous seminorm. By Theorem 4.4.13, there is compact $K \subseteq X$ and $M > 0$ so that

$$|\varphi(f)| \leq M \sup_K |f|$$

for all $f \in C(X)$.

Let $e : K \to X$ be the embedding map. This gives rise to the continuous restriction $e^* : C(X) \to C(K)$. We will show that e^* is a surjection, that is any continuous $f : K \to \mathbb{K}$ may be extended to X.

Let $i : K \to \mathrm{tych}(X)$ be the domain restriction of the quotient map. We will argue that i is an embedding. Suppose $x, y \in K$ are distinct. Since X is functionally Hausdorff due to being c-embedded, we have that the constant net at x does not converge to y. There is then some continuous $f : X \to \mathbb{K}$ which distinguishes between x and y. Thus, $i(x) \neq i(y)$ and we have that i is an injection. Further, it is the restriction of the quotient map and thus continuous. Lastly, since K is a compact c-embedded space, it is topological Corollary 3.4.13. The codomain restriction of i is then a continuous surjection from a compact topological space to a Hausdorff space, so that it is a homeomorphism by Proposition 2.6.36.

We thus may view K as a subset of $\mathrm{tych}(X)$. Since it is a compact subset of a Tychonoff space, any continuous $f : K \to \mathbb{K}$ may be extended to a continuous $\widehat{f} : X \to \mathbb{K}$ (see Appendix C.3). We may then continue this to X by composing with the quotient $q : X \to \sigma(X)$; that is, $\widehat{f} \circ q : X \to \mathbb{K}$ extends f for any $f \in C(K)$. In other words, we have that $C(K) = \{f|_K : f \in C(X)\}$.

Observe that if $g, f \in C(X)$ agree on K, then

$$|\varphi(f - g)| \leq M \sup_K |f - g| = 0$$

so that $\varphi(f) = \varphi(g)$. Thus, we may safely define $\widetilde{\varphi} : C(K) \to \mathbb{K}$ by $\widetilde{\varphi}(f|_K) = \varphi(f)$. Certainly, we know that

$$\varphi = \widetilde{\varphi} \circ e^* = (e^*)'(\widetilde{\varphi})$$

so that it only remains to show that $\widetilde{\varphi}$ is continuous. Suppose we have net Λ in $C(X)$ so that $\Lambda|_K \to f|_K$ for some $f \in C(X)$. This is equivalent to $(\Lambda - f)|_K \to 0$. We now have

$$|\varphi(\Lambda - f)| \leq M \sup_K |\Lambda - f| = M \sup |(\Lambda - f)|_K| \to 0$$

so that $\varphi(\Lambda) \to \varphi(f)$ and thus, $\widetilde{\varphi}(\Lambda|_K) \to \widetilde{\varphi}(f|_K)$. Therefore, $\widetilde{\varphi}$ is continuous which concludes the proof. QED

4.6 Reflexivity

In this section we define reflexive convergence vector spaces and prove that the paradual of any convergence space is reflexive.

Lemma 4.6.1. *If V is a CVS, then map $j_V : V \to \mathcal{L}_c\,\mathcal{L}_c(V)$ given by $j_V(v) = \mathrm{ev}_v$ is well defined, linear, and continuous.*

Proof. Linearity of j_V clear as is the linearity of $\mathrm{ev}_v : \mathcal{L}_c(V) \to \mathbb{K}$ for any $v \in V$. We next check that $j_V(v)$ is continuous. Suppose we have a net $\Lambda \to f$ in $\mathcal{L}_c(V)$. We have that $j_V(v)(\Lambda) = \Lambda(v) \to f(v) = j_V(v)(f)$. Thus, $j_V(v)$ is continuous for each $v \in V$.

We lastly check that j_V is itself continuous. This follows by Corollary 3.2.11 since its primary transpose is $\mathrm{ev} : V \times \mathcal{L}_c(V) \to \mathbb{K}$ which is continuous. QED

Notation 4.6.2. If V and W are CVSs and $f : V \to W$ is linear and continuous, then the adjoint of the adjoint of f is denoted by $f'' : \mathcal{L}_c\,\mathcal{L}_c(V) \to \mathcal{L}_c\,\mathcal{L}_c(W)$; that is by $f'' = (f')'$.

Remark 4.6.3. A straightforward diagram chase shows that if V and W are CVSs and $f : V \to W$ is linear and continuous, then

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & W \\ {\scriptstyle j_V}\big\downarrow & & \big\downarrow{\scriptstyle j_W} \\ \mathcal{L}_c\,\mathcal{L}_c(V) & \xrightarrow{\quad f'' \quad} & \mathcal{L}_c\,\mathcal{L}_c(W) \end{array}$$

commutes. It is often convenient to write $\widehat{v}$ for $j_V(v) = \mathrm{ev}_v$. The commutativity of the above diagram then reads as $f''(\widehat{v}) = \widehat{f(v)}$ for all $v \in V$.

Definition 4.6.4. If V is a CVS, then V is called *reflexive* when j_V is an isomorphism, i.e. a linear homeomorphism.

Proposition 4.6.5. *If V and W are isomorphic CVSs, then V is reflexive if and only if W is.*

Proof. Let $h : V \to W$ be an isomorphism. It follows that h'' is an isomorphism. Suppose j_V is an isomorphism. Observe that $j_W = h'' \circ j_V \circ h^{-1}$ is a composition of isomorphisms and thus an isomorphism. QED

The last result of this section will be that paraduals are reflexive. We will first need two lemmas.

Lemma 4.6.6. *If V is a convergence vector space, a subset $B \subseteq \mathcal{L}(V)$ is called* equicontinuous *when for any $\epsilon > 0$ there exists for each $v \in V$ a vicinity U of v so that for all $b \in B$ and $w \in U$ one has $|b(w) - b(v)| < \epsilon$. If B is equicontinuous, then the weak* and continuous subspace convergence structures on B are identical.*

Proof. Let B_σ and B_c denote B with its weak* and continuous convergence respectively. By Corollary 4.5.7 we know that the identity map $B_c \to B_\sigma$ is continuous and so it is left to demonstrate that the inverse is continuous. Suppose $f \in B$ and Λ is a net in B with $\Lambda \to f$ in B_σ. Suppose α is a net in V with $\alpha \to v$ for some $v \in V$.

Fix $\epsilon > 0$. There is some vicinity U of v so that that for all $b \in B$ and $w \in U$ one has $|b(w) - b(v)| < \epsilon$. In particular, for each $\ell \in \mathrm{dom}(\Lambda)$ one has $|\Lambda_\ell(w) - \Lambda_\ell(v)| < \epsilon$ for each $w \in U$. Since $\alpha \to v$, we have that $\alpha \in_{\mathrm{ev}} U$. Thus, for $i \in \mathrm{dom}(\alpha)$ large enough and any $\ell \in \mathrm{dom}(\Lambda)$ there holds $|\Lambda_\ell(\alpha_i) - \Lambda_\ell(v)| < \epsilon/2$. Since $\Lambda \to f$ in the weak* sense, we have that $\Lambda(v) \to f(v)$. Thus, when $\ell \in \mathrm{dom}(\Lambda)$ is large enough, we have $|\Lambda_\ell(v) - f(v)| < \epsilon/2$. It follows by application of the triangle equality that for ℓ, i large enough $|\Lambda_\ell(\alpha_i) - \Lambda_f(v)| < \epsilon$. Therefore, $\Lambda(\alpha) \to f(v)$. As this holds for all $v \in V$ and nets $\alpha \to v$ in V, we have that $\Lambda \to f$ in B_c as desired. QED

Lemma 4.6.7. *If V is a normed space and B denotes the closed unit ball in $\mathcal{L}_{\mathrm{op}}(V)$, then B is equicontinuous.*

Proof. Fix $\epsilon > 0$ and $1, \epsilon > \delta > 0$. Let $v \in V$ and $w \in B(v, \delta)$ and $\varphi \in B$. We then have $||w - v|| < \epsilon$ so that $||(w - v)/\epsilon|| < 1$. We compute

$$\begin{aligned} |\varphi(w) - \varphi(v)| &= \epsilon|\varphi(w - v)/\epsilon| \\ &\leq \epsilon||\varphi|| && \text{(definition of operator norm)} \\ &< \epsilon \end{aligned}$$

so that B is equicontinuous. QED

Theorem 4.6.8. *If X is a convergence space, then $C_c(X)$ is reflexive.*

Proof. Recall from Definition 3.4.1 the map $i_X : X \to C_c C_c(X)$. We restrict the codomain and redefine to obtain $i_X : X \to \mathcal{L}_c\, C_c(X)$. Next, we consider i_X^* and restrict the domain to obtain $i_X^* : \mathcal{L}_c\, \mathcal{L}_c\, C_c(X) \to C_c(X)$. Observe that if $f \in C_c(X)$ and $x \in X$, then

$$\begin{aligned} i_X^*(j_{C_c(X)}(f))(x) &= j_{C_c(X)}(f) \circ i_X(x) \\ &= i_X(x)(f) \\ &= f(x). \end{aligned}$$

Thus, $i_X^* \circ j_{C_c(X)} = \mathrm{id}_{C_c(X)}$. To show that i_X^* is also a right inverse for $j_{C_c(X)}$, it suffices to show that i_X^* is an injection.

Let V denote the subspace of $\mathcal{L}_c\, C_c(X)$ spanned by $i_X(X)$. Suppose that V is dense in $\mathcal{L}_c\, C_c(X)$. Suppose there are $\varphi, \psi \in \mathcal{L}_c\, \mathcal{L}_c\, C_c(X)$ so that $i_X^*(\varphi) = i_X^*(\psi)$. Let $f \in \mathcal{L}_c\, C_c(X)$. We may find a net α in V so that $\alpha \to f$. By continuity, we have that $\varphi(\alpha) \to \varphi(f)$ and $\psi(\alpha) \to \psi(f)$. Observe that for any index a for α, we may find

$\lambda_1, ..., \lambda_n \in \mathbb{K}$ and $x_1, ..., x_n \in X$ so that

$$\begin{aligned}\varphi_(\alpha_a) &= \varphi\Bigg(\sum_{i=1}^{n} \lambda_i i_X(x_i)\Bigg) \\ &= \sum_{i=1}^{n} \lambda_i i_X^*(\varphi)(x_i) \\ &= \sum_{i=1}^{n} \lambda_i i_X^*(\psi)(x_i) \\ &= \psi\Bigg(\sum_{i=1}^{n} \lambda_i i_X(x_i)\Bigg) \\ &= \psi(\alpha_a)\end{aligned}$$

so that $\varphi(\alpha) = \psi(\alpha)$. Since $\mathbb{K}$ is Hausdorff, we have that $\varphi(f) = \psi(f)$ and thus $\varphi = \psi$. We conclude that to show that i_X^* is injective, it suffices to show that V is dense in $\mathcal{L}_c\, C_c(X)$.

We first consider the case wherein X is a compact topological convergence space. In this case, we have by Corollary 4.2.5 that $C_c(X)$ is a Banach space under the supremum norm. Let $||\cdot|| : C_c(X) \to \mathbb{R}$ denote the supremum norm on $C_c(X)$ and let $||\cdot|| : \mathcal{L}\, C_c(X) \to \mathbb{R}$ be the usual operator norm

$$||\varphi|| = \sup\{|f(x)| : ||x|| \leq 1\}.$$

Define

$$B = \{\varphi \in \mathcal{L}_c\, C_c(X) : ||\varphi|| \leq 1\}$$

and

$$E = \{\alpha i_X(x) : \alpha \in \mathbb{K} \ \wedge \ |\alpha| = 1 \ \wedge \ x \in X\}.$$

It is known[7] that E is the set of extreme points of B. Further, B is compact by the Banach-Alaoglu theorem.[8] Thus, the Krein-Milman theorem[9] tells us that $B = \overline{\operatorname{co}(E)}$ where this closure is taken with respect to the weak*-topology. By Lemma 4.6.6 and Lemma 4.6.7, be have that $B = \overline{\operatorname{co}(E)} = a(\operatorname{co}(E))$ in the continuous convergence structure as well. Thus, if $\varphi \in \mathcal{L}_c\, C_c(X)$, either $\varphi = 0 \in V$ or there is a net α in $\operatorname{co}(E) \subseteq V$ so that $\alpha \to \varphi/||\varphi||$ and $||\varphi||\alpha \to \varphi$ in the continuous convergence structure. Since $||\varphi||\alpha$ is a net in V, we then have that V is dense if $\mathcal{L}_c\, C_c(X)$ as desired.

[7] See Lemma V.8.6 of [Dun58]

[8] See, for instance, Theorem 8.4.1 of [NB10].

[9] See, for instance, Theorem 9.4.6 of [NB10].

We proceed to the case in which X may not be compact. Since $C_c(X) \cong C_c(c(X))$, we may assume that X is c-embedded. Every compact $K \subseteq X$ is c-embedded and thus topological and we note that

$$\begin{array}{ccc} K & \xhookrightarrow{\quad i \quad} & X \\ {\scriptstyle i_K}\downarrow & & \downarrow{\scriptstyle i_X} \\ \mathcal{L}_c\, C_c(K) & \xrightarrow[\quad i^{**} \quad]{} & \mathcal{L}_c\, C_c(X) \end{array}$$

commutes. If $A \subseteq \mathcal{L}_c\, C_c(X)$ define $[A]$ to be the subspace of $\mathcal{L}_c\, C_c(X)$ spanned by A. We see that

$$\begin{aligned} \mathrm{cl}([i_X(X)]) &\supseteq \mathrm{cl}([i_X(i(K))]) \\ &= \mathrm{cl}([i^{**}(i_K(K))]) \\ &\supseteq i^{**}(\mathrm{cl}([i_K(K)])) \\ &= i^{**}(\mathcal{L}_c\, C_c(K)) \end{aligned}$$

so that

$$\mathrm{cl}([i_X(X)]) \supseteq \bigcup \{ i^{**}(\mathcal{L}_c\, C_c(K)) : K \subseteq X \text{ compact} \}.$$

This right hand side is exactly $\mathcal{L}_c\, C_c(X)$ by Proposition 4.5.24. Therefore, $V = [i_X(X)]$ is dense in $\mathcal{L}_c\, C_c(X)$ which establishes that i_X^* is an injection. This in turn implies that i_X^* is right inverse to $j_{C_c(X)}$. Since we have already shown this to be a left inverse, we have that $j_{C_c(X)}$ is an embedding and $C_c(X)$ is reflexive. QED

Conclusion

Here ends this introduction to convergence spaces, introducing basic properties and constructions in this setting. It has been shown that convergence spaces extend the notion of topological spaces and capture types of convergence which are fundamentally non-topological. Hopefully the reader has seen the benefits offered by convergence spaces, including a natural and painless proof of Tychonoff's theorem and a canonical convergence structure for function spaces.

It hardly need be said that there is much more to the study of convergence spaces than is contained in this thesis. In his PhD dissertation [Pat14], Patten explores a natural convergence structure on reflexive digraphs and extends the notion of differentials from single variable calculus to the broader setting of convergence spaces. The text [DM16] by Dolecki and Mynard offers an introduction to topology from the viewpoint of convergence spaces and delves much further than this work into convergence theory. Of much interest to this author is the extension of the idea of completeness of metric spaces to the larger setting of convergence spaces where it remains closely tied to compactness. The functional analytic side of convergence theory can be explored more deeply in [BB02]. Here one finds, amongst other fascinating topics, convergence theoretic versions of the Hahn-Banach and Banach-Steinhaus theorems.

Appendix A

Some Category Theory

The language of category theory is used sporadically throughout this work. This appendix gathers together some basic terms and results so that a reader unfamiliar with category theory need not find another source. Sections one and two draw from [Lei14] while sections three and four are based on Chapters 4 and 5 of [BBT20].

A.1 What Are Categories?

Definition A.1.1. A category **C** consists of the following data

1. A collection $\mathrm{ob}\,\mathbf{C}$ called the *objects* of **C**. We write $A \in \mathbf{C}$ for $A \in \mathrm{ob}\,\mathbf{C}$.

2. For each $A, B \in \mathbf{C}$ a collection $\mathrm{Hom}(A, B)$ of *morphisms*. If $f \in \mathrm{Hom}(A, B)$, we say that the *source* of f is A and B is the *target* of f. One writes $f : A \to B$ to mean $f \in \mathrm{Hom}(A, B)$. Two morphisms of **C** may only be equal if they have the same source and target.

3. For each $A, B, C \in \mathbf{C}$ a function

$$\circ : \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$$

   which is associative, that is if $A, B, C, D \in \mathbf{C}$ and $f : A \to B$ and $g : B \to C$ and $h : C \to D$ then $h \circ (g \circ f) = (h \circ g) \circ f$.

4. For each $A \in \mathbf{C}$ a morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ so that for all $B \in \mathbf{C}$, all $f : B \to A$, and all $g : A \to B$ that $\mathrm{id}_A \circ f = f$ and $g \circ \mathrm{id}_A = g$.

Example A.1.2. **SET** is the category with

1. objects: sets,

2. for sets A, B the morphisms $\mathrm{Hom}(A, B)$ are the functions from A to B,

3. composition: the usual composition of functions.

Example A.1.3. **TOP** is the category with

1. objects: topological spaces,
2. for topological spaces A, B the morphisms $\mathrm{Hom}(A, B)$ are the continuous functions from A to B,
3. composition: the usual composition of functions.

Example A.1.4. If G is a group, one may consider the category with

1. objects: a single object X,
2. $\mathrm{Hom}(X, X) = G$,
3. If $g, h \in \mathrm{Hom}(X, X)$ then $g \circ h = gh$ is the product of g and h in G.

Example A.1.5. If **C** is a category, one may consider the *opposite category* $\mathbf{C}^{\mathrm{op}}$

1. objects: $\mathrm{ob}\,\mathbf{C}$,
2. $\mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathbf{C}}(Y, X)$.
3. So that, denoting composition in **C** with $\circ$ and in $\mathbf{C}^{\mathrm{op}}$ by $*$, we have $f * g = g \circ f$.

Definition A.1.6. If **C** is a category with objects A, B, a morphism $f : A \to B$ is called an *isomorphism* when there exists a morphism $g : B \to A$ so that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. When there exists an isomorphism between object $A, B \in \mathbf{C}$, one says A and B are isomorphic and writes $A \cong B$.

Definition A.1.7. A category **C** is called *locally small* when for each $X, Y \in \mathbf{C}$ the collection $\mathrm{Hom}(X, Y)$ is a set.

Definition A.1.8. A category is called *small* when it is locally small and the collection of objects is a set.

Definition A.1.9. Suppose $\mathcal{A}$ is a family of objects[10] in a category **C**. The *categorical product* of $\mathcal{A}$, if it exists, is an object $\prod A$ of **C** along with morphisms $\pi_A : \prod \mathcal{A} \to A$ for each $A \in \mathcal{A}$ so that for any object Ω of **C** with morphisms $f_A : \Omega \to A$ for each $A \in \mathcal{A}$ there is a unique morphism $h : \Omega \to \prod \mathcal{A}$ so that

$$\begin{array}{ccc} \Omega & \xrightarrow{\;h\;} & \prod \mathcal{A} \\ & \searrow^{f_A} & \downarrow \pi_A \\ & & A \end{array}$$

commutes for each $A \in \mathcal{A}$.

[10] Note that this family may be of objects with indices so that a single object may appear more than once.

Notation A.1.10. If $\mathcal{A} = \{A_1, A_2, ..., A_n\}$, write $A_1 \times A_2 \times \cdots \times A_n$ for $\prod \mathcal{A}$.

Example A.1.11. In **SET**, the cartesian product is the categorical product. In **TOP**, the cartesian product equipped with the product topology is the categorical product.

Definition A.1.12. Suppose $\mathcal{A}$ is a family of objects in a category **C**. The *categorical coproduct* of $\mathcal{A}$, if it exists, is an object $\coprod A$ of **C** along with morphisms $e_A : A \to \coprod \mathcal{A}$ for each $A \in \mathcal{A}$ so that for any object Ω of **C** with morphisms $f_A : A \to \Omega$ for each $A \in \mathcal{A}$ there is a unique morphism $h : \coprod \mathcal{A} \to \Omega$ so that

$$\begin{array}{ccc} \coprod \mathcal{A} & \xrightarrow{h} & \Omega \\ {\scriptstyle e_A}\uparrow & \nearrow_{f_A} & \\ A & & \end{array}$$

commutes for each $A \in \mathcal{A}$.

Notation A.1.13. If $\mathcal{A} = \{A_1, A_2, ..., A_n\}$, write $A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$ for $\coprod \mathcal{A}$.

Remark A.1.14. The universal property of products and coproducts determines these objects up to isomorphism, that is any two objects satisfying either Definition A.1.9 or Definition A.1.12 (for the same collection of objects) must be isomorphic.

A.2 Functors, Natural Transformations, and Equivalence of Categories

Definition A.2.1. If **C** and **D** are categories, a *functor* $F : \mathbf{C} \to \mathbf{D}$ consists of assignments

1. for each $A \in \mathbf{C}$ an object $F(A) \in \mathbf{D}$, and
2. for each $A, B \in \mathbf{C}$ and morphism $f : A \to B$ a morphism $F(f) : F(A) \to F(B)$

so that

1. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for each $A \in \mathbf{C}$, and
2. for each $A, B, C \in \mathbf{D}$ and functions $f : A \to B$ and $f : B \to C$ that $F(g \circ f) = F(g) \circ F(f)$.

Remark A.2.2. Each category **C** has an *identity functor* $\mathrm{id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ which does nothing to objects or morphisms.

Definition A.2.3. A functor $F : \mathbf{C} \to \mathbf{D}$ is called *faithful* (resp. *full*) when for all $A, B \in \mathbf{C}$, the assignment $\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ given by $f \mapsto F(f)$ is injective (resp. surjective).

Definition A.2.4. A functor $F : \mathbf{C} \to \mathbf{D}$ is called *essentially surjective on objects* if for all $B \in \mathbf{D}$ there is $A \in \mathbf{C}$ so that $F(A) \cong B$.

Definition A.2.5. If $\mathbf{C}$ and $\mathbf{D}$ are categories and $F, G : \mathbf{C} \to \mathbf{D}$ are functors, a *natural transformation* α from F to G consists of the following data:

1. for each object $A \in \mathbf{C}$ a morphism $\alpha_A : F(A) \to G(A)$ in $\mathbf{D}$ called the *component of* α *at* A

2. so that for all objects $A, B \in \mathbf{C}$ and all morphism $f : A \to B$, the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle\alpha_A}\downarrow & & \downarrow{\scriptstyle\alpha_B} \\ G(A) & \xrightarrow[G(f)]{} & G(B) \end{array}$$

   commutes. Two natural transformations between the same functors are called equal when their component morphisms are equal.

Remark A.2.6. It is not difficult to verify that the composition of natural transformations via the composition of their components results in another natural transformation. Further, each functor has an identity natural transformation, *i.e.* one whose components are each identity morphisms. Thus, for categories $\mathbf{C}$ and $\mathbf{D}$, one may consider the functor category $[\mathbf{C}, \mathbf{D}]$ whose objects are functors from $\mathbf{C}$ to $\mathbf{D}$ and whose morphisms are natural transformations.

Lemma A.2.7. *A natural transformation α is a (natural) isomorphism of functors $F, G : \mathbf{C} \to \mathbf{D}$ if and only if each component α_A is an isomorphism for each $A \in \mathbf{C}$.*

Proof. Suppose α is an isomorphism. We then have that there is some natural transformation α^{-1} from G to F so that $\alpha\alpha^{-1} = id_G$ and $\alpha^{-1}\alpha = id_F$. However, we know the components of a composition are just the composition of components and the components of the identity natural transformation are simply the identity maps. That is, for each $A \in \mathbf{C}$, we have $\alpha_A\alpha_A^{-1} = id_{G(A)}$ and $\alpha_A^{-1}\alpha_A = id_{F(A)}$. Thus, each component of α is an isomorphism.

Now, suppose that for each component α_A of α, we have an inverse α_A^{-1}. It suffices to show that the α_A^{-1} form the components of a natural transformation from G to F. To this end, let $f : A \to B$ be a morphism in $\mathbf{C}$. Indeed, it follows from

$$G(f) \circ \alpha_A = \alpha_B \circ F(f)$$

and the existence of α_A^{-1} and α_B^{-1} that

$$\alpha_B^{-1} G(f) = F(f)\alpha_A^{-1}$$

which is naturality of the transformation with components α_A^{-1} for each $A \in \mathbf{C}$. QED

Definition A.2.8. Two categories **C** and **D** are called *equivalent* when there exist functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $\alpha : F \circ G \to id_{\mathbf{C}}$ and $\beta : G \circ F \to id_{\mathbf{D}}$.

Theorem A.2.9. *Categories* **C** *and* **D** *are equivalent if and only if there exists a functor* $F : \mathbf{C} \to \mathbf{D}$ *which is full, faithful, and essentially surjective on objects.*

Proof. Suppose **C** and **D** are equivalent categories. We produce functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $\alpha : F \circ G \to id_{\mathbf{D}}$ and $\beta : id_{\mathbf{C}} \to G \circ F$. We will prove that F is full, faithful, and essentially surjective on objects.

Suppose $B \in \mathbf{D}$. We then have that $G(B) \in \mathbf{C}$ and $F \circ G(B) \in \mathbf{D}$. Further, $\alpha_B : F \circ G(B) \to B$ is an isomorphism since α is a natural isomorphism. Thus, F is essentially surjective on objects.

Now, suppose that $f, f' : A \to A'$ are morphisms in **C** so that $F(f) = F(f')$. Thus, $G \circ F(f) = G \circ F(f')$. Consider the following two naturality squares glued together:

$$\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & A' \\
{\scriptstyle \beta_A}\downarrow & & \downarrow{\scriptstyle \beta_{A'}} \\
G \circ F(A) & \xrightarrow{G \circ F(f)} & G \circ F(A') \\
{\scriptstyle \beta_A^{-1}}\downarrow & & \downarrow{\scriptstyle \beta_{A'}^{-1}} \\
A & \xrightarrow[\quad f' \quad]{} & A'
\end{array}$$

This whole diagram commutes, from which one may read $f = f'$. Thus, F is faithful. By analogous reasoning, G is faithful.

Lastly, let $A, B \in \mathbf{C}$ and $g \in \mathrm{Hom}(F(A), F(A'))$. We define $f : A \to B$ by $f = \beta_B^{-1} G(g) \beta_A$. By naturality, we have that

$$\begin{array}{ccc}
G \circ F(A) & \xrightarrow{G \circ F(f)} & G \circ F(B) \\
{\scriptstyle \beta_A}\uparrow & & \uparrow{\scriptstyle \beta_B} \\
A & \xrightarrow[\quad f \quad]{} & B
\end{array}$$

commutes. From this, one may determine that $G(g) = G \circ F(f)$. By faithfulness of G, we have $g = F(f)$. Thus, F is full.

Now, we proceed to the other direction. Let $F : \mathbf{C} \to \mathbf{D}$ be full, faithful, and essentially surjective on objects. We will construct $G : \mathbf{D} \to \mathbf{C}$ witnessing the equivalence of **C** and **D**. By the essential injectivity on objects of F, for each $B \in \mathbf{D}$, we may find some $G(B) \in \mathbf{C}$ so that $F(G(B)) \cong B$ with isomorphism witnessed by γ_B. Now, supposing that $B, B' \in \mathbf{D}$ and there is some map $f : B \to B'$, one considers the composition:

$$\gamma_{B'}^{-1} f \gamma_B : F(G(B)) \to F(G(B')).$$

By fullness of F, we have some $G(f) : G(B) \to G(B')$ so that $F(G(f)) = \gamma_{B'}^{-1} f \gamma_B$. We claim the object and morphism assignments given by $B \mapsto G(B)$ and $f \mapsto G(f)$ define a functor $G : \mathbf{D} \to \mathbf{C}$. Let $B \in \mathbf{D}$. We have that $F(G(id_B)) = id_{F(G(B))}$, and by faithfulness of F that $G(id_B) = id_{G(B)}$. A similar process and invocation of faithfulness also shows that G respects composition.

We now need only show that $F \circ G$ and $G \circ F$ satisfy the appropriate naturality condition. We claim for each $B \in \mathcal{B}$ that γ_B is the component of a natural isomorphism $\gamma : F \circ G \to id_{\mathcal{B}}$. That each component is an isomorphism has already been established. The commutation of the appropriate naturality square is given purely from the definition of G applied to morphisms.

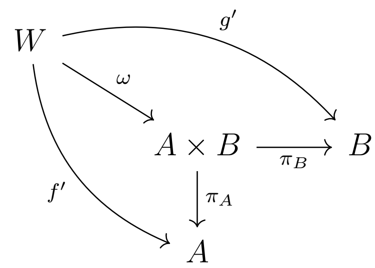
Lastly, we must find a natural isomorphism $\alpha : G \circ F \to id_{\mathbf{C}}$. Suppose $A \in \mathbf{C}$. One then has $\gamma_{F(A)} : F(G \circ F(A)) \to F(A)$. By fullness of F, we may find $\alpha_A : G \circ F(A) \to A$ so that $F(\alpha_A) = \gamma_{F(A)}$. We claim that this defines the components of the natural isomorphism α. Note that α_A is invertible as there is $\beta : A \to G \circ F(A)$ so that $F(\beta) = \gamma_{F(A)}^{-1}$ (by fullness of F), so that $F(\beta\alpha_A) = F(\alpha_A\beta) = id_{F(A)}$ which implies $\beta = \alpha_A^{-1}$ by faithfulness of F. It remains to show that α is natural. Let $A, A' \in \mathcal{A}$ and $f : A \to A'$. Consider the naturality square

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & A' \\ {\scriptstyle \alpha_A}\uparrow & & \uparrow{\scriptstyle \alpha_{A'}} \\ G \circ F(A) & \xrightarrow[G \circ F(f)]{} & G \circ F(A') \end{array}$$

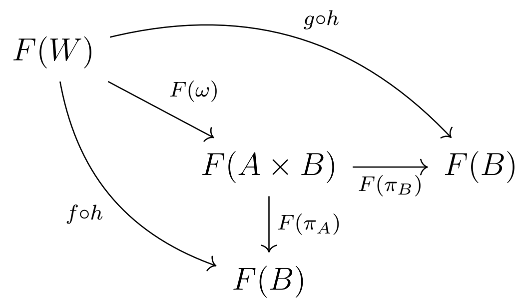
which commutes by definition of α and faithfulness of F. QED

Proposition A.2.10. *Suppose* $\mathbf{C}$ *and* $\mathbf{D}$ *are categories and* $F : \mathbf{C} \to \mathbf{D}$ *is a full and essentially surjective on objects functor. For any objects* $A, B \in \mathbf{C}$ *so that* $A \times B$ *exists, then* $F(A) \times F(B)$ *exists and* $F(A \times B) \cong F(A) \times F(B)$*.*

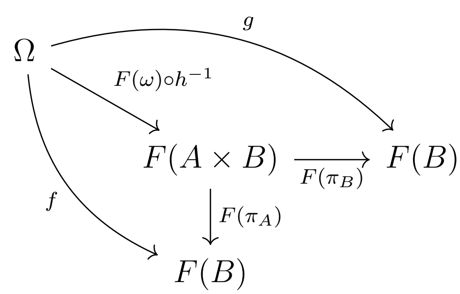
Proof. We have that $F(\pi_A) : F(A \times B) \to F(A)$ and $F(\pi_B) : F(A \times B) \to F(B)$. Suppose that Ω is an object of $\mathbf{D}$ with $f : \Omega \to A$ and $g : \Omega \to B$. Since F is essentially surjective on objects, we have some $W \in \mathbf{C}$ so that $\Omega \cong F(W)$. Let $h : F(W) \to \Omega$ witness this isomorphism. We have that $f \circ h : F(W) \to F(A)$ and $g \circ h : F(W) \to F(B)$. By fullness of F, there is some $f' : W \to A$ and $g' : W \to B$ so that $F(f') = f \circ h$ and $F(g') = g \circ h$. By the universal property governing products in $\mathbf{C}$, there is some $\omega : W \to A \times B$ so that



commutes. It follows that



commutes. From this we conclude that



commutes. Thus, $F(A \times B) \cong F(A) \times F(B)$ QED

Definition A.2.11. If **C** and **D** are categories, we say that **D** is a *subcategory* of **C** when the collection of objects and morphisms of **D** is a subcollection of the objects of **C** and the inclusion $\mathbf{D} \to \mathbf{C}$ is a functor.

A.3 Limits and Colimits

Definition A.3.1. A *diagram* in a category **C** is a functor $D : \mathbf{J} \to \mathbf{C}$ where **J** is a small category. One says that the diagram D is **J** *shaped*. Given an object $A \in \mathbf{C}$ one has access to the *constant J shaped diagram* $\mathbf{J}_A$ which sends each object of **J** to A and each morphism to id_A.

Definition A.3.2. Given a diagram $D : \mathbf{J} \to \mathbf{C}$ and an object $A \in \mathbf{C}$, a *cone from A to D* is a natural transformation $\mathbf{J}_A \to D$ and a *cone from D to A* is a natural transformation $D \to \mathbf{J}_A$.

Definition A.3.3. A *limit* of a diagram $D : \mathbf{J} \to \mathbf{C}$ is an object $\lim D$ of **C** along with a cone $\eta : \lim D \to D$ so that for all $A \in \mathbf{C}$ and cones $\gamma : A \to D$ there exists a unique morphism $h : A \to \lim D$ so that for all $J \in \mathbf{J}$ the diagram

$$\begin{array}{ccc} A & & \\ {\scriptstyle h}\big\downarrow & \overset{\gamma_J}{\searrow} & \\ \lim D & \xrightarrow[\eta_J]{} & DJ \end{array}$$

commutes. Likewise, a *colimit* of D is an object $\operatorname{colim} D$ of **C** along with a cone $\epsilon : D \to \operatorname{colim} D$ so that for all $A \in$ **C** cones $\lambda : D \to A$ there exists a unique morphism $h : \operatorname{colim} D \to A$ so that for all $J \in$ **J** the diagram

$$\begin{array}{ccc} DJ & & \\ {\scriptstyle \epsilon_J}\big\downarrow & \overset{\lambda_J}{\searrow} & \\ \operatorname{colim} D & \xrightarrow[h]{} & A \end{array}$$

commutes.

Proposition A.3.4. *If $S : \mathbf{J} \to \mathbf{C}$ is a diagram with limit $\lim D$ and $A \in \mathbf{C}$ is isomorphic to $\lim D$, then A is a limit of D.*

Proof. Let $f : A \to \lim D$ be an isomorphism. Let $\eta : \lim D \to D$ a cone witnessing that $\lim D$ is a limit. For each $J \in$ **J**, define $\epsilon_J : A \to DJ$ by $\epsilon_J = \eta_J f$. One sees that this defines a cone $\epsilon : A \to D$. Suppose $\gamma : B \to D$ is a cone. We observe that

$$\begin{array}{ccc} B & & \\ {\scriptstyle f^{-1}h}\big\downarrow & \overset{\gamma_J}{\searrow} & \\ A & \xrightarrow[\epsilon_J]{} & DJ \end{array}$$

commutes where h is the unique morphism making

$$\begin{array}{ccc} B & & \\ {\scriptstyle h}\big\downarrow & \overset{\gamma_J}{\searrow} & \\ \lim D & \xrightarrow[\eta_J]{} & DJ \end{array}$$

commute. Should any other h' make

$$\begin{array}{ccc} B & & \\ {\scriptstyle h'}\big\downarrow & \overset{\gamma_J}{\searrow} & \\ A & \xrightarrow[\epsilon_J]{} & DJ \end{array}$$

commute, then

$$\begin{array}{ccc} B & & \\ {\scriptstyle fh'}\big\downarrow & \overset{\gamma_J}{\searrow} & \\ \lim D & \xrightarrow[\eta_J]{} & DJ \end{array}$$

commutes so that $fh' = h$ and $h' = f^{-1}h$. This show that A is a limit of D. QED

Proposition A.3.5. *Limits and colimits of diagrams are unique up to isomorphism.*

Proof. Suppose **C** is a category and $D : \mathbf{J} \to \mathbf{C}$ is some diagram. Suppose A and B are limits of D with witnessing cones $\alpha : A \to D$ and $\beta : B \to D$. There are then unique morphisms $h_1 : A \to B$ and $h_2 : B \to A$ so that for all $J \in \mathbf{J}$ we have that the diagrams

$$
\begin{array}{ccc}
A & & \\
{\scriptstyle h_1}\downarrow & \searrow^{\alpha_J} & \\
B & \xrightarrow[\beta_J]{} & DJ
\end{array}
\qquad\qquad
\begin{array}{ccc}
B & & \\
{\scriptstyle h_2}\downarrow & \searrow^{\beta_J} & \\
A & \xrightarrow[\alpha_J]{} & DJ
\end{array}
$$

commute. But then we see that the diagrams

$$
\begin{array}{ccc}
A & & \\
{\scriptstyle \mathrm{id}_A}\downarrow & \searrow^{\alpha_J} & \\
A & \xrightarrow[\alpha_J]{} & DJ
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & & \\
{\scriptstyle h_2 h_1}\downarrow & \searrow^{\alpha_j} & \\
A & \xrightarrow[\alpha_J]{} & DJ
\end{array}
$$

commute. Therefore, by the uniqueness clause in the definition of limit, we have $h_2 h_1 = \mathrm{id}_A$. By like reasoning we have $h_1 h_2 = \mathrm{id}_B$ so that h_1 and h_2 are isomorphisms.

The proof for colimits is similar. QED

Remark A.3.6. It is thus sensible to talk about *the* (co)limit of a diagram.

Example A.3.7. Let **C** be a category. Let $\mathcal{A}$ be any small subcategory with only identity morphisms. Let $D : \mathcal{A} \to \mathbf{C}$ be given by $D(A) = A$. Then, if it exists, $\lim D = \prod \mathcal{A}$ and $\operatorname{colim} D = \coprod \mathcal{A}$.

Definition A.3.8. The limit of a diagram of the shape

$$\bullet \rightrightarrows \bullet$$

is called an *equalizer*. The colimit of such a diagram is called a *coequilizer*.

Definition A.3.9. A category is called *complete* if it contains limits for all its diagrams. A category is called *cocomplete* if it has colimits for all its diagrams.

Theorem A.3.10. *A category is complete if it has all equalizers and small products (those taken over sets). A category is cocomplete when it has all coequilizers and all small coproducts.*

Proof. Suppose **C** is a category with all small products and all equalizers and $D : \mathbf{J} \to \mathbf{C}$ is a diagram. Define

$$A = \prod_{J \in \mathbf{J}} DJ.$$

Thus, for each $J \in \mathbf{J}$ we have projections $\pi_J^A : A \to DJ$. Let $\mathrm{Hom}(\mathbf{J})$ denote the set of morphisms in $\mathbf{J}$ and for each morphism $f \in \mathrm{Hom}(\mathbf{J})$, denote by $s(f)$ the domain object of f and by $t(f)$ the image object of f. Define

$$B = \prod_{f \in \mathrm{Hom}(\mathbf{J})} Dt(f).$$

We have then for each $f \in \mathrm{Hom}(\mathbf{J})$ projection $\pi_f^B : B \to Dt(f)$. Observe that for each $f \in \mathrm{Hom}(\mathbf{J})$ we have a morphism $Df \circ \pi_{s(f)}^A : A \to Dt(f)$. We also have $\pi_{t(f)}^A : A \to Dt(f)$. Thus, by the universal property of products, we have unique maps $h, i : A \to B$ so that for each $f \in \mathrm{Hom}(\mathbf{J})$ the diagrams

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ & \searrow^{Df \circ \pi^A_{Ds(f)}} & \downarrow \pi_f^B \\ & & Dt(f) \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{i} & B \\ & \searrow^{\pi^A_{Dt(f)}} & \downarrow \pi_f^B \\ & & Dt(f) \end{array}$$

commute. Let Ω be the equalizer of

$$A \overset{h}{\underset{i}{\rightrightarrows}} B$$

with $\omega : \Omega \to A$ the given morphism with $h\omega = i\omega$. We claim that Ω is a limit of D. For each $J \in \mathbf{J}$, define $\eta_J : \Omega \to DJ$ by $\eta_J = \pi_J^A \circ \omega$. We then see that for all morphism $f : J \to J'$ in $\mathbf{J}$, the diagram

$$\begin{array}{ccc} & \Omega & \\ {}^{\eta_J}\swarrow & & \searrow^{\eta_{J'}} \\ DJ & \xrightarrow[Df]{} & DJ' \end{array}$$

commutes since

$$\begin{aligned} Df \circ \eta_J &= Df \circ \pi_J^A \circ \omega \\ &= \pi_f^B \circ h \circ \omega \\ &= \pi_f^B \circ i \circ \omega \\ &= \pi_{J'}^A \circ \omega \\ &= \eta_{J'}. \end{aligned}$$

This shows that the η_J are components of a cone $\eta : \Omega \to D$. Now, suppose $\gamma : W \to D$ is another cone. By the universal property of products there is a map $s : W \to A$ so that for all $J \in \mathbf{J}$

$$\begin{array}{ccc} W & \xrightarrow{s} & A \\ & \searrow^{\gamma_J} & \downarrow \pi_J^A \\ & & DJ \end{array}$$

commutes. Thus, we have for all $f \in \text{Hom}(J)$ that

$$\begin{aligned} \pi_f^B \circ h \circ s &= Df \circ \gamma_{s(f)} \\ &= \gamma_{t(f)} \end{aligned}$$

and

$$\begin{aligned} \pi_f^B \circ i \circ s &= \pi_{t(f)}^A \circ s \\ &= \gamma_{t(f)}. \end{aligned}$$

So, by the uniqueness clause in the definition of products, we have that $h \circ s = i \circ s$. Therefore, W has a cone over the diagram for which Ω is an equalizer. We thus have a unique map $r : W \to \Omega$ so that

$$W \xrightarrow{r} \Omega \xrightarrow{\omega} A \underset{i}{\overset{h}{\rightrightarrows}} B$$

commutes and $\omega \circ r = s$.

We then have that for all $J \in \mathbf{J}$ the diagram

$$\begin{array}{ccc} W & & \\ {\scriptstyle r}\downarrow & \searrow^{\gamma_J} & \\ \Omega & \xrightarrow[\eta_J]{} & DJ \end{array}$$

commutes.

Lastly, suppose $r' : W \to \Omega$ so that

$$\begin{array}{ccc} W & & \\ {\scriptstyle r'}\downarrow & \searrow^{\gamma_J} & \\ \Omega & \xrightarrow[\eta_J]{} & DJ \end{array}$$

commutes. We then have that $\omega \circ r' = s$ by the universal property constructing s. But since Ω is the equalizer, this forces $r = r'$.

This shows that Ω is a limit of D as desired.

The proof for cocompleteness is similar. QED

Definition A.3.11. A functor $F : \mathbf{C} \to \mathbf{D}$ is called *continuous* when for all diagrams D in $\mathbf{C}$ with a limit $\lim D$, one has $F(\lim D)$ is a limit of FD. Likewise, F is called *cocontinuous* when for all diagrams D in $\mathbf{C}$ with a colimit $\operatorname{colim} D$, one has $F(\operatorname{colim} D)$ is a colimit of FD.

A.4 Adjunctions

Throughout this section, all categories will be assumed to be locally small.

Lemma A.4.1. *Suppose* **C** *and* **D** *are categories and* $L : \mathbf{C} \to \mathbf{D}$ *and* $R : \mathbf{D} \to \mathbf{C}$ *are a pair of functors. Fix an object* X *of* **C** *and* Y *of* **D***. The following are functors:*

1. $\mathrm{Hom}(X, R-) : \mathbf{D} \to \mathbf{SET}$ *defined for objects* $Z, Z' \in \mathbf{D}$ *with morphism* $f : Z \to Z'$ *as*

$$\mathrm{Hom}(X, Rf) : \mathrm{Hom}(X, RZ) \to \mathrm{Hom}(X, RZ') \quad \text{by} \quad g \mapsto R(f) \circ g$$

2. $\mathrm{Hom}(LX, -) : \mathbf{D} \to \mathbf{SET}$ *defined for objects* $Z, Z' \in \mathbf{D}$ *with morphism* $f : Z \to Z'$ *as*

$$\mathrm{Hom}(LX, f) : \mathrm{Hom}(LX, Z) \to \mathrm{Hom}(LX, Z') \quad \text{by} \quad g \mapsto f \circ g$$

3. $\mathrm{Hom}(-, RY) : \mathbf{C}^{\mathrm{op}} \to \mathbf{SET}$ *defined for objects* $Z, Z' \in \mathbf{C}$ *with morphism* $f : Z \to Z'$ *as*

$$\mathrm{Hom}(f, RY) : \mathrm{Hom}(Z', RY) \to \mathrm{Hom}(Z, RY) \quad \text{by} \quad g \mapsto g \circ f$$

4. $\mathrm{Hom}(L-, Y) : \mathbf{C}^{\mathrm{op}} \to \mathbf{SET}$ *defined for objects* $Z, Z' \in \mathbf{C}$ *with morphism* $f : Z \to Z'$ *as*

$$\mathrm{Hom}(Lf, Y) : \mathrm{Hom}(Z', RY) \to \mathrm{Hom}(Z, RY) \quad \text{by} \quad g \mapsto g \circ L(f)$$

Definition A.4.2. Suppose **C** and **D** are categories. A pair of functors $L : \mathbf{C} \to \mathbf{D}$ and $R : \mathbf{D} \to \mathbf{C}$ is called an *adjunction* when for each object $X \in \mathbf{C}$ and $Y \in \mathbf{D}$ there is an isomorphism of sets

$$\lambda_{X,Y} : \mathrm{Hom}(LX, Y) \cong \mathrm{Hom}(X, RY)$$

so that for each $X \in \mathbf{C}$ the family

$$\lambda_{X,-} : \mathrm{Hom}(LX, -) \to \mathrm{Hom}(X, R-)$$

and for each $Y \in \mathbf{D}$ the family

$$\lambda_{-,Y} : \mathrm{Hom}(L-, Y) \to \mathrm{Hom}(-, RY)$$

is a natural transformation. In this case, L is called *left adjoint* and R is called *right* adjoint.

There is another characterization of adjoints given by the next theorem.

Theorem A.4.3. *An adjunction between categories* **C** *and* **D** *is a pair of functor* $L : \mathbf{C} \to \mathbf{D}$ *called the* left adjoint *and* $R : \mathbf{D} \to \mathbf{C}$ *called the* right adjoint *along with natural transformation* $\eta : id_{\mathbf{C}} \to RL$ *and* $\epsilon : LR \to id_{\mathbf{D}}$ *call the* unit *and* counit of adjunction *so that for all* $X \in \mathbf{C}$ *and* $Y \in \mathbf{D}$ *the diagrams*

$$\begin{array}{ccc} LX & \xrightarrow{L\eta_x} & LRLX \\ & \searrow^{id_{LX}} & \downarrow \epsilon_{LX} \\ & & LX \end{array} \qquad \begin{array}{ccc} RY & \xrightarrow{\eta_{RY}} & RLRY \\ & \searrow^{id_{RY}} & \downarrow R\epsilon_{Y} \\ & & RY \end{array}$$

commute.

Proposition A.4.4. *Suppose $L : \mathbf{C} \to \mathbf{D}$ and $R : \mathbf{D} \to \mathbf{C}$ are left and right adjoints. Then L is cocontinuous and R is continuous.*

Definition A.4.5. A functor $R : \mathbf{D} \to \mathbf{C}$ is said to satisfy the *solution set conditions* when for each object $A \in \mathbf{C}$ there exists a set $\{B_i\}_{i \in I}$ of objects in $\mathbf{D}$ and a set of morphisms $\{f_i : A \to RB_i\}_{i \in I}$ so that for any $B \in \mathbf{D}$ and morphism $f : A \to RB$ there is $i \in I$ and $h : B_i \to B$ so that

$$\begin{array}{ccccc} A & & \xrightarrow{\quad f \quad} & & RB \\ & \searrow_{f_i} & & \nearrow_{Rh} & \\ & & RB_i & & \end{array}$$

Theorem A.4.6 (Freyd's Adjoint Functor Theorem)**.** *If D is a locally small, complete category and $R : \mathbf{D} \to \mathbf{C}$ is a continuous functor satisfying the solution set conditions, then R has a left adjoint.*

For a proof of this result, see Chapter 4.6 of [Rie16].

Definition A.4.7. A (locally small) category $\mathbf{C}$ is called Cartesian closed, when for each $C \in \mathbf{C}$ the functor $(\cdot) \times C : \mathbf{C} \to \mathbf{C}$ has a right adjoint.

Appendix B

Net-Filter Equivalence and Convergence as Functor

In this appendix, we describe the equivalence of nets and filters in the language of category theory. We then give an alternative definition of convergence space where the convergence structure is a functor and an equivalent definition where nets are the primitive converging objects. As far as we know this characterization does not appear in the literature and is completely new.

B.1 Net-Filter Equivalence

A key conceptual motif of Chapter 1 is that nets and filters are, in some sense, the same thing. Using category theory, we can make this equivalence very precise.

Definition B.1.1. Given a set X we define two categories.

1. The first is the *filter category of* X, denoted $\Phi(X)$.
   (i) The objects of $\Phi(X)$ are filters on X.
   (ii) If $\mathcal{F}, \mathcal{G}$ are filters on X, we set
   $$\mathrm{Hom}(\mathcal{F},\mathcal{G}) = \begin{cases} (\mathcal{F},\mathcal{G}) & \mathcal{F} \subseteq \mathcal{G} \\ \emptyset & \text{else} \end{cases}$$
   (iii) Given morphisms $(\mathcal{F},\mathcal{G})$ and $(\mathcal{G},\mathcal{H})$, we define $(\mathcal{F},\mathcal{G}) \circ (\mathcal{G},\mathcal{H}) = (\mathcal{F},\mathcal{H})$.
2. The second is the *net category of* X, denoted $\mathfrak{N}(X)$.
   (i) The objects of $\mathfrak{N}(X)$ are nets in X.
   (ii) If α, β are nets in X, we set
   $$\mathrm{Hom}(\alpha,\beta) = \begin{cases} (\alpha,\beta) & \beta \text{ is a subnet of } \alpha \\ \emptyset & \text{else} \end{cases}$$

(iii) Given morphisms (α, β) and (β, γ), we define $(\alpha, \beta) \circ (\beta, \gamma) = (\alpha, \gamma)$.

with these definitions in place, we may consider the formation of eventuality filters and derived nets as functors between the filter and net categories.

Definition B.1.2. For any set X, we define functors

1. $\eta : \Phi(X) \to \mathfrak{N}(X)$ by $\mathcal{F} \mapsto \eta(\mathcal{F})$, and if $(\mathcal{F}, \mathcal{G})$ is a morphism in $\Phi(X)$, then $(\mathcal{F}, \mathcal{G}) \mapsto (\eta(\mathcal{F}), \eta(\mathcal{G}))$.
2. $\mathcal{E} : \mathfrak{N}(X) \to \Phi(X)$ by $\alpha \mapsto \mathcal{E}(\alpha)$, and if (α, β) is a morphism in $\mathfrak{N}(X)$, then $(\alpha, \beta) \mapsto (\mathcal{E}(\alpha), \mathcal{E}(\beta))$.

Remark B.1.3. These functors are certainly well defined on objects and are well defined on morphisms by Definition 1.3.1 (1) and Theorem 1.3.4.

Theorem B.1.4. *For any set X, the categories $\mathfrak{N}(X)$ and $\Phi(X)$ are equivalent.*

Proof. By Theorem A.2.9, it suffices to check that $\mathcal{E} : \mathfrak{N}(X) \to \Phi(X)$ is full, faithful, and essentially surjective on objects.

Since for any filter $\mathcal{F}$ on X we have $\mathcal{E}(\eta(\mathcal{F})) = \mathcal{F}$ by Theorem 1.2.14, it follows that $\mathcal{E}$ is full.

Next, fix nets α, β in X. If $\mathrm{Hom}(\alpha, \beta) = \emptyset$, then β is not a subnet of α and so by definition $\mathcal{E}(\beta) \not\subseteq \mathcal{E}(\alpha)$ and so $\mathrm{Hom}(\mathcal{E}(\alpha), \mathcal{E}(\beta)) = \emptyset$ too. Otherwise, we have β is a subnet of α. By definition, we have $\mathcal{E}(\alpha) \subseteq \mathcal{E}(\beta)$ and $\mathrm{Hom}(\mathcal{E}(\alpha), \mathcal{E}(\beta)) = (\mathcal{E}(\alpha), \mathcal{E}(\beta))$ while $\mathrm{Hom}(\alpha, \beta) = (\alpha, \beta)$. We then see that $\mathcal{E}$ is full.

Lastly, we have that $\mathcal{E}$ is faithful since each hom-set in $\mathfrak{N}(X)$ contains at most one morphism.

We conclude that $\mathfrak{N}(X)$ and $\Phi(X)$ are equivalent. QED

B.2 Convergence as Functor

Definition B.2.1. If X is a set, a *filter convergence structure* on X is a functor $\lambda : \Phi(X) \to \mathcal{P}(X)$ such that

1. for all $x \in X$, we have $x \in \lambda([x])$;
2. for all $\mathcal{F}, \mathcal{G} \in \Phi(X)$ we have$\lambda(\mathcal{F}) \cap \lambda(\mathcal{G}) \subseteq \lambda(\mathcal{F} \cap \mathcal{G})$

Remark B.2.2. Condition (2) may be reworded as follows. First, note that in light of functoriality we have for any filter $\mathcal{F}, \mathcal{G}$ on X that

$$\lambda(\mathcal{F} \cap \mathcal{G}) \subseteq \lambda(\mathcal{F}) \cap \lambda(\mathcal{G}).$$

Thus, in fact, condition (2) is equivalent to

$$\lambda(\mathcal{F}) \cap \lambda(\mathcal{G}) = \lambda(\mathcal{F} \cap \mathcal{G}).$$

Further, finite limits in both $\Phi(X)$ and $\mathcal{P}(X)$ are just intersection. So, property 2 can by rephrased as λ preserves finite limits.

The idea here us that λ assigns to each filter its set of limits. Indeed, it is not difficult to see that this notion of filter convergence structure is identical to that already in use. This new formulation does not immediately offer any benefit over the usual definitions of convergence structure. The first payoff is that a similar definition may be given for a net convergence structure without running into size issues with the class of nets. It is then possible to prove that the corresponding notions of convergence are identical.

Definition B.2.3. If X is a set, we define a *net convergence structure* to be a functor $\lambda : \mathfrak{N}(X) \to \mathcal{P}(X)$ such that

1. If α is a constant net in X with value x, then $x \in \lambda(\alpha)$;
2. for each $\alpha, \beta \in \mathfrak{N}(X)$ we have $\lambda(\alpha) \cap \lambda(\beta) \subseteq \lambda(\alpha \wedge \beta)$.

Theorem B.2.4. *Let X be any set.*

1. *If λ_1 is a filter convergence structure, then $\lambda_1 \circ \mathcal{E} : \mathfrak{N}(X) \to \mathcal{P}(X)$ is a net convergence structure and every net convergence structure arises this way.*
2. *If λ_2 is a net convergence structure, then $\lambda_2 \circ \eta : \Phi(X) \to \mathcal{P}(X)$ is a filter convergence structure and all filter convergence structures arise this way.*

Proof. We prove the first half of each statement first. Certainly, $\lambda_1 \circ \mathcal{E} : \mathfrak{N}(X) \to \mathcal{P}(X)$ is a functor with the proper source and target categories. Now, suppose that α is a net in X with constant value x. We then have that $\mathcal{E}(\alpha) = [x]$. Since λ_1 is a filter convergence structure, then $x \in \lambda_1[x]$. We conclude that $x \in \lambda_1 \circ \mathcal{E}(\alpha)$. Next, supposing that α, β are nets in X, we have that $\mathcal{E}(\alpha \wedge \beta) = \mathcal{E}(\alpha) \cap \mathcal{E}(\beta)$. Since λ_1 is a filter convergence structure, we have that $\lambda_1(\mathcal{E}(\alpha)) \cap \lambda_1(\mathcal{E}(\beta)) \subseteq \lambda_1(\mathcal{E}(\alpha) \cap \mathcal{E}(\beta))$. This is then exactly that $\lambda_a \circ \mathcal{E}(\alpha) \cap \lambda_1 \circ \mathcal{E}(\beta) \subseteq \lambda_1 \circ \mathcal{E}(\alpha \wedge \beta)$. Thus, $\lambda_1 \circ \mathcal{E}$ is a net convergence structure.

A nearly identical argument holds for $\lambda_2 \circ \eta$. We of course have that this is a functor with the proper source and target categories. If $x \in X$, we have that $\eta([x]) = \alpha_{[x]} \sim \alpha$ where α is any net with constant value x. We have since λ_2 is a functor that $\lambda_2(\eta([x])) \cong \lambda_2(\alpha)$. However, in $\mathcal{P}(X)$ the only isomorphisms are equalities, so $\lambda_2(\eta[x]) = \lambda_2(\alpha)$. Thus, since λ_2 is a filter convergence structure, $x \in \lambda_2 \circ \eta([x])$. Next, suppose that $\mathcal{F}, \mathcal{G}$ are filters on X. We have that $\eta(\mathcal{F} \cap \mathcal{G}) = \eta(\mathcal{F}) \wedge \eta(\mathcal{G})$. Thus,

$$\lambda_2 \circ \eta(\mathcal{F}) \cap \lambda_2 \circ \eta(\mathcal{G}) \subseteq \lambda_2 \circ \eta(\mathcal{F} \cap \mathcal{G}).$$

Thus, $\lambda_2 \circ \eta$ is a net convergence structure.

Now, suppose that λ is any filter convergence structure. We claim that $\lambda = \lambda \circ \mathcal{E} \circ \eta$. This is clear since $\mathcal{E} \circ \eta = \mathrm{id}_{\Phi(X)}$.

Lastly, suppose that λ is any net convergence structure. We then claim that $\lambda = \lambda \circ \eta \circ \mathcal{E}$. This is certainly true. If α is any net, then $\alpha \sim \eta \circ \mathcal{E}(\alpha)$. Thus, $\lambda(\alpha) \cong \lambda \circ \eta \circ \mathcal{E}(\alpha)$. But, again, the only isomorphisms in $\mathcal{P}(X)$ are equalities. So, $\lambda(\alpha) = \lambda \circ \eta \circ \mathcal{E}(\alpha)$ QED

B.2.1 Continuity

Let X and Y be sets. A function $f : X \to Y$ induces two functors $f_\Phi : \Phi(X) \to \Phi(Y)$ and $f_\mathcal{P} : \mathcal{P}(X) \to \mathcal{P}(Y)$. These are given by $f_\Phi(\mathcal{F}) = f(\mathcal{F})$ and $f_\mathcal{P}(A) = f(A)$ for any filter $\mathcal{F}$ on X and any $A \subseteq X$.

Theorem B.2.5. *Let X and Y be convergence spaces. Let λ_X and λ_Y be the filter convergence structures inherent to X and Y. A function $f : X \to Y$ is continuous if and only if there exists a natural transformation from $f_\mathcal{P} \circ \lambda_X$ to $\lambda_Y \circ f_\Phi$.*

Proof. Suppose such a natural transformation A exists. For any filter $\mathcal{F}$ on X, we then have a morphism from $f_\mathcal{P} \circ \lambda_X(\mathcal{F})$ to $\lambda_Y \circ f_\Phi(\mathcal{F})$. Of course, this just means $f_\mathcal{P} \circ \lambda_X(\mathcal{F}) \subseteq \lambda_Y \circ f_\Phi(\mathcal{F})$. Suppose that $\mathcal{F} \to x$ in X. We then have $x \in \lambda_X(\mathcal{F})$ and $f(x) \in f_\mathcal{P} \circ \lambda_X(\mathcal{F})$. Thus, $f(x) \in \lambda_Y \circ f_\Phi(\mathcal{F})$, so $f(\mathcal{F}) \to f(x)$. This is exactly continuity of f.

Now, suppose that f is continuous. Suppose $\mathcal{F}$ is a filter on X. By continuity, if $\mathcal{F} \to x$, then $f(\mathcal{F}) \to f(x)$. This is exactly that $f_\mathcal{P} \circ \lambda_X(\mathcal{F}) \subseteq \lambda_Y \circ f_\Phi(\mathcal{F})$. Thus, if $\mathcal{F} \subseteq \mathcal{G}$ are filters on X we have that

$$\begin{array}{ccc} f_\mathcal{P} \circ \lambda_X(\mathcal{F}) & \longrightarrow & f_\mathcal{P} \circ \lambda_X(\mathcal{G}) \\ \downarrow & & \downarrow \\ \lambda_Y \circ f_\Phi(\mathcal{F}) & \longrightarrow & \lambda_Y \circ f_\Phi(\mathcal{G}) \end{array}$$

commutes. This is the desired naturality diagram. QED

Appendix C

Some Topological Results

This appendix contains various results from topology for the reader's convenience.

C.1 Normal Spaces, Urysohn's Lemma, and Partitions of Unity

The results of this section are standard, though many of the proof techniques used here are inspired by the discussion on the relevant nLab pages.

Definition C.1.1. A topological space X is called *normal* when for every pair $A, B \subseteq X$ of disjoint, closed subsets of X there are open sets $U \supseteq A$ and $V \supseteq B$ so that $U \cap V = \emptyset$.

Proposition C.1.2. *If X is a normal space and $Y \subseteq X$ is closed, then Y with the subspace topology is normal.*

Proof. Let $A, B \subseteq Y$ be closed and disjoint. We have closed subsets $A^*, B^* \subseteq X$ so that $A = A^* \cap Y$ and $B = B^* \cap Y$. Since Y is closed in X, it follows that A and B are closed in X. We may thus find $U \supseteq A$ and $U \supseteq B$ open in X which are disjoint. Thus, $U \cap Y \supseteq A$ and $V \cap Y \supseteq B$ are open and disjoint in Y. QED

Definition C.1.3. The *dyadic rationals* are $\{\frac{a}{2^n} : a \in \mathbb{Z} \ \wedge \ n \in \mathbb{N}\}$.

Remark C.1.4. The dyadic rationals are dense in $\mathbb{Q}$. We will write $\mathcal{Q}$ for the dyadic rationals.

Theorem C.1.5. *A topological space X is normal iff for all closed $C \subseteq X$ and open sets $U \supseteq C$ there exists open V such that*

$$C \subseteq V \subseteq \overline{V} \subseteq U.$$

Proof. Suppose X is normal. Let $C \subseteq U \subseteq X$ with C closed and U open. We have that $X \setminus U$ is closed. Since X is normal, we have that there exists open sets $U_C \supseteq C$ and $U_{X \setminus U} \supseteq X \setminus C$ such that

$$U_C \cap U_{X \setminus U} = \emptyset.$$

It follows that $U_C \subseteq X \setminus U_{X\setminus U}$. Since $U_{X\setminus U}$ is open, we have $X \setminus U_{X\setminus U}$ is closed. Therefore,

$$C \subseteq U_C \subseteq \overline{U_C} \subseteq X \setminus U_{X\setminus U} \subseteq U$$

where this last containment follows from $X \setminus U \subseteq U_{X\setminus U}$. Eliding one of the sets in the above yields

$$C \subseteq U_C \subseteq \overline{U_C} \subseteq U$$

which is precisely the desired result.

We proceed to the other direction. Suppose that for all closed $C \subseteq X$ and open sets $U \supseteq C$ there exists open V such that

$$C \subseteq V \subseteq \overline{V} \subseteq U.$$

Suppose C_1, C_2 are disjoint closed subsets of X. We have that $X \setminus C_2$ is open. We may then find open V such that

$$C_1 \subseteq V \subseteq \overline{V} \subseteq X \setminus C_2.$$

We then have that $C_1 \subseteq V$ and $C_2 \subseteq X \setminus \overline{V}$ which is open. It is immediately clear that $V \cap (X \setminus \overline{V}) = \emptyset$. Thus, we may separate any two disjoint closed subsets of X by open sets, that is X is normal. QED

Theorem C.1.6. ***(Urysohn's Lemma)*** *Suppose X is a topological space. Then X is normal iff for any disjoint, closed $A, B \subseteq X$, there exists continuous $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Proof. We will start with the simple direction of showing that X is normal if for any disjoint, closed $A, B \subseteq X$, there exists continuous $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Let A, B be closed, disjoint subsets of X and $f : X \to [0,1]$ have the aforementioned properties. We have that $A \subseteq f^{-1}([0, \frac{1}{2}))$ and $B \subseteq f^{-1}((\frac{1}{2}, 1])$. Since $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$ are open and disjoint in $[0,1]$, the continuity of f guarantees that their preimages are open and disjoint in X. Thus, X is normal.

We now proceed to the harder direction of the proof. Assume X is a normal space. Define $C_0 = A$ and $U_1 = X \setminus B$. Since $A \cap B = \emptyset$, we see that $C_0 \subseteq U_1$. By Theorem C.1.5 we may find an open set $U_{1/2}$ and closed set $C_{1/2}$ such that

$$C_0 \subseteq U_{1/2} \subseteq C_{1/2} \subseteq U_1.$$

This process may be iterated to obtain closed sets $C_{1/4}, C_{3/4}$ and open sets $U_{1/4}, U_{3/4}$ such that

$$C_0 \subseteq U_{1/4} \subseteq C_{1/4} \subseteq U_{1/2} \subseteq C_{1/2} \subseteq U_{3/4} \subseteq C_{3/4} \subseteq U_1.$$

This process may be iterated indefinitely to obtain a collection of open sets

$$\mathcal{U} = \{U_r : r \in \mathcal{Q} \cap (0,1]\}$$

and a collection of closed sets

$$\mathcal{C} = \{C_r : r \in \mathcal{Q} \cap (0,1]\}$$

such that if $r, s \in \mathcal{Q}$ and $r < s$,

$$U_r \subseteq C_r \subseteq U_s \subseteq C_s.$$

We the define $f : X \to [0,1]$ by

$$f(x) = \begin{cases} \inf\{r \in \mathcal{Q} \cap (0,1] : x \in U_r\} & x \in U_1 \\ 1 & x \in B \end{cases}$$

We observe that $f(B) = \{1\}$ by definition and $f(A) = \{0\}$ since the sequence $(1/n) \subseteq \mathcal{Q} \cap (0,1]$. It only remains to show that f is continuous.

We note that because

$$\mathcal{B} = \{[0,\alpha), (\alpha, 1] : \alpha \in (0,1)\}$$

is a subbase for the metric topology on $[0,1]$, we need only show that $f^{-1}((\alpha,1])$ and $f^{-1}([0,\alpha))$ are open in X for any $\alpha \in [0,1]$. We claim that

$$f^{-1}((\alpha,1]) = \bigcup_{r>\alpha} (X \setminus C_r).$$

Suppose $x \in f^{-1}((\alpha,1])$. Thus $\alpha < f(x)$. Indeed, this means we may[11] find a dyadic rational s such that $\alpha < s < f(x)$. Indeed, we may find a second dyadic rational r such that

$$\alpha < r < s < f(x).$$

From this, one obtains $x \notin U_s$

$$r < f(x) = \inf\{r \in \mathcal{Q} \cap [0,1] : x \in U_r\}$$

implies $s \notin \{r \in \mathcal{Q} \cap [0,1] : x \in U_r\}$. It follows that $x \notin C_r$. Therefore $x \in X \setminus C_r$ with $r > \alpha$. We obtain

$$f^{-1}((\alpha,1]) \subseteq \bigcup_{r>\alpha} (X \setminus C_r).$$

[11]This follows from the dyadic rationals being dense in $\mathbb{Q}$.

To check the other containment, let $x \in X \setminus C_r$ for some dyadic $r > \alpha$. We have $x \notin U_r$. Then $f(x) \geq r$. This demonstrates the other containment and we obtain

$$f^{-1}((\alpha, 1]) = \bigcup_{r>\alpha}(X \setminus C_r).$$

This is a union of open sets and thus is open.

Next, we claim that

$$f^{-1}([0, \alpha)) = \bigcup_{r<\alpha} U_r.$$

Suppose $x \in f^{-1}([0, \alpha))$. Then $f(x) < \alpha$. There is then dyadic r such that $f(x) < r < \alpha$. This means $x \in U_r$. These steps run backwards to prove

$$f^{-1}([0, \alpha)) = \bigcup_{r<\alpha} U_r.$$

Again, this is open as a union of open sets.

We know then that f is continuous and conclude the proof. QED

Proposition C.1.7. *Compact Hausdorff topological spaces are normal.*

Proof. Let X be a compact and Hausdorff topological space with closed, disjoint subsets A and B. Note that A and B are compact. For each $a \in A$ and $b \in B$ find disjoint open sets $U_{a,b} \ni a$ and $V_{a,b} \ni b$. For each $b \in B$, the collection $\mathcal{U}_b = \{U_{a,b} : a \in A\}$ is an open cover for A and thus admits a finite subcover $\mathcal{U}'_b$. Define

$$V_b = \bigcap\{V_{a,b} : a \in A \text{ and } V_{a,b} \in \mathcal{U}'_b\}$$

which is an open neighborhood of b and disjoint from each element of $\mathcal{U}'_b$. We have that $\mathcal{V} = \{V_b : b \in B\}$ is an open cover of B and thus admits finite subcover $\mathcal{V}'$. Define

$$U = \bigcap\left\{\bigcup \mathcal{U}'_b : V_b \in \mathcal{V}'\right\}.$$

We see that $A \subseteq U$ which is open. We also have that $B \subseteq V$ where $V = \bigcup \mathcal{V}'$ which is open. We last note that $V \cap U = \emptyset$. Therefore, X is normal. QED

Lemma C.1.8. *If X is a normal space with finite open cover $\{U_i\}_{i=1}^n$ then there exists an finite open cover $\{V_i\}_{i=1}^n$ so that for each $i = 1, ..., n$ we have*

$$V_i \subseteq cl(V_i) \subseteq U_i.$$

Proof. If $n = 1$, this follows from Theorem C.1.5. Suppose $n = 2$. That is, we have an open cover of the form $\{U_1, U_2\}$. We have that $X \smallsetminus U_1$ and $X \smallsetminus U_2$ are disjoint and closed. By normality, we have disjoint open sets V_1, V' so that $X \smallsetminus U_2 \subseteq V_1$ and $X \smallsetminus U_1 \subseteq V'$. We observe that

$$V_1 \subseteq X \smallsetminus V' \subseteq U_1$$

from which it follows from closedness of $X \smallsetminus V'$ that

$$V_1 \subseteq \text{cl}(V_1) \subseteq U_1.$$

Observe that $\{V_1, U_2\}$ is still an open cover of X. Thus, we may repeat the above process to obtain open $V_2 \subseteq U_2$ with $\{V_1, V_2\}$ an open cover of X and

$$V_2 \subseteq \text{cl}(V_2) \subseteq U_2.$$

Now, suppose $n > 2$. Consider the cover $\{U_1, \bigcup_{k=2}^n U_k\}$. Repeat the above process to find open $V_1 \subseteq U_1$ with the desired properties. Then split off each other U_k in turn and do the same. QED

Definition C.1.9. If X is a topological space, a *partition of unity* on X is a collection $\{u_i\}_{i \in I}$ of continuous functions $u_i : X \to [0, 1]$ so that

1. For each $x \in X$ there are only finitely many $i \in I$ so that $u_i(x) \neq 0$;
2. For each $x \in X$ we have $\sum_{i \in I} u_i(x) = 1$.

Further, if $\{U\}_{i \in I}$ is an open cover of X and for each $i \in I$ we have supp$\{u_i\} \subseteq U_i$, then we say the partition is *subordinate* to the open cover.

Proposition C.1.10. *If X is a compact, Hausdorff topological space and $\{U_i\}_{i \in I}$ is an open cover of X, then there exists a partition of unity on X which is subordinate to $\{U_i\}_{i \in I}$.*

Proof. Let $J \subseteq I$ be a finite subset of I so that $\{U_j\}_{j \in J}$ is an open cover of X. Since compact, Hausdorff spaces are normal, we may find covers $\{W_j\}_{j \in J}$ and $\{V_j\}$ so that

$$W_j \subseteq \text{cl}(W_j) \subseteq V_j \subseteq \text{cl}(V_j) \subseteq U_j$$

for each $j \in J$. For each $j \in J$ we have disjoint closed sets cl(W_j) and $X \smallsetminus V_j$. Since X is normal, Urysohn's lemma guarantees a continuous function $h_j : X \to [0, 1]$ which takes 0 on cl(W_j) and 1 on $X \smallsetminus V_j$. We observe that $h_j^{-1}(\{0\}) \subseteq V_j$ and thus supp$(h_j) \subseteq$ cl$(V_j) \subseteq U_j$. For each $i \in I \smallsetminus J$ define $h_i : X \to [0, 1]$ to be constantly zero. Define $N : X \to [0, 1]$ by

$$N(x) = \sum_{i \in I} h_i(x).$$

Observe that this is well defined since at most finitely many of the h_i may be non-zero and that $N(x) \neq 0$ for all $x \in X$. Further, N is continuous as a sum of continuous functions all but finitely many of which are uniquely 0. We next define for each $i \in I$ the continuous function $u_i : X \to [0, 1]$ by $u_i = \frac{h_i}{N}$. We finally observe that $\{u_i\}_{i \in I}$ is a partition of unity subordinate to $\{U_i\}_{i \in I}$. QED

C.2 The Stone-Čech Compactification

This section proves the existence of the Stone-Čech Compactification and establishes some of its basic properties. For the most part, these results here are more detailed versions of the discussion of the Stone-Čech compactification in [BBT20].

Definition C.2.1. Let **CHAUS** denote the category of compact Hausdorff topological spaces and continuous maps. There is an inclusion functor U : **CHAUS** $\to$ **TOP**. This functor has a left adjoint β : **TOP** $\to$ **CHAUS** called the *Stone-Čech compactification*.

It is not immediately clear that U has a left adjoint. The following results will set up an existence proof for β by means of Theorem A.4.6 - Freyd's adjoint functor theorem.

Lemma C.2.2. *The category of topological spaces is complete.*

Proof. By Theorem A.3.10 it suffices to show that **TOP** has all (small) products and all equalizers. Certainly **TOP** is closed under products. Additionally, it is not hard to verify that the equalizer of a diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

in **TOP** is $\{x \in X : f(x) = g(x)\}$ with the subspace topology. QED

Corollary C.2.3. *The category of compact Hausdorff topological spaces is complete.*

Proof. Since products of compact Hausdorff spaces are compact and Hausdorff, we have that **CHAUS** is closed under (small) products. Further, if

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

is a diagram in **CHAUS**, one has that $\{x \in X : f(x) = g(x)\}$ is closed in X and thus compact and Hausdorff when equipped with the subspace topology. So **CHAUS** has all (small) products and equalizes and is therefore complete. QED

Corollary C.2.4. *The inclusion functor* U : **CHAUS** $\to$ **TOP** *is continuous.*

Lemma C.2.5. *The inclusion functor* U : **CHAUS** $\to$ **TOP** *satisfies the solution set conditions layed out in Definition A.4.5.*

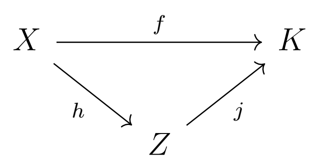
Proof. Fix a topological space X. If K is compact and Hausdorff and $f : X \to K$, then $\text{cl}(f(X))$ is compact and Hausdorff. Further, we have

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & K \\ & {\scriptstyle \widehat{f}}\searrow \quad \nearrow{\scriptstyle i} & \\ & \text{cl}(f(X)) & \end{array}$$

commutes when $\widehat{f}$ is the codomain restriction of f and i is the subset inclusion map. Observe that since each element of $\mathrm{cl}(f(X))$ can be witnessed by a distinct converging filter, this has cardinality at most $|P|$ where $P = \mathcal{P}(\mathcal{P}(X))$. Construct a set $\mathcal{A}$ in the following way:

- Take all subsets of P.
- Determine all compact and Hausdorff topological spaces whose underlying set is a subset of P.
- Include each into $\mathcal{A}$.

Define $\mathcal{B}$ to be the collection of all continuous functions from X into elements of $\mathcal{A}$. With these constructions, there will be $Z \in \mathcal{A}$ with $Z \cong \mathrm{cl}(f(X))$. Using this homeomorphism, there will be $h : X \to Z$ and some continuous $j : Z \to K$ so that

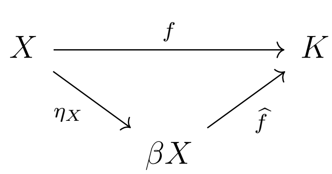


commutes. These are the desired solution set conditions. QED

Theorem C.2.6. *The inclusion functor U : **CHAUS** $\to$ **TOP** has a left adjoint.*

Proof. With the preceding lemmas, we safely invoke the Freyd adjoint functor theorem. QED

Proposition C.2.7. *For every topological space X, there is a map $\eta_X : X \to \beta X$ so that for all compact, Hausdorff K and continuous $f : X \to K$ there is a unique continuous map $\widehat{f} : \beta X \to K$ so that*



commutes.

Proof. Fix a topological space X. For all compact, Hausdorff space Y, we have an isomorphism

$$h_{X,Y} : C(\beta X, Y) \to C(X, Y)$$

which is natural in Y. Define $\eta_X : X \to \beta X$ by $\eta_X = h_{X,\beta X}(\mathrm{id}_{\beta X})$. Suppose K is some compact Hausdorff space and $f : X \to Y$ is continuous. Define $\widehat{f} : \beta X \to K$ by $\widehat{f} = h^{-1}_{X,K}(f)$. Naturality in Y tells us that

$$\begin{array}{ccc} C(\beta X, \beta X) & \xrightarrow{C(\beta X, \widehat{f})} & C(\beta X, K) \\ {\scriptstyle h_{X,\beta X}}\big\downarrow & & \big\downarrow{\scriptstyle h_{X,K}} \\ C(X, \beta X) & \xrightarrow[C(X,\widehat{f})]{} & C(X, K) \end{array}$$

commutes. If we trace $\mathrm{id}_{\beta X}$ through both both upper and lower halves of the diagram, we get

$$\widehat{f} \circ \eta_X = h_{X,K}(\widehat{f}) = f$$

which is exactly what it means for the desired diagram to commute.

It is left to show that $\widehat{f}$ is unique. Suppose there is a continuous map g making

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & K \\ & {\scriptstyle \eta_X}\searrow \quad \nearrow{\scriptstyle g} & \\ & \beta X & \end{array}$$

commutes. We then have by naturality that

$$\begin{array}{ccc} C(\beta X, \beta X) & \xrightarrow{C(\beta X, g)} & C(\beta X, K) \\ {\scriptstyle h_{X,\beta X}}\big\downarrow & & \big\downarrow{\scriptstyle h_{X,K}} \\ C(X, \beta X) & \xrightarrow[C(X,g)]{} & C(X, K) \end{array}$$

commutes. Again, this means

$$h_{X,K}(g) = g \circ \eta_X = f.$$

Since $h_{X,K}$ is a bijection, $g = \widehat{f}$ so that $\widehat{f}$ is indeed unique. QED

Definition C.2.8. A topological space X is called *Tychonoff* when it is Hausdorff and for every $x \in X$ and closed $A \subseteq X \setminus \{x\}$ there is a continuous $f : X \to [0,1]$ so that $f(x) = 1$ and $f(A) = \{0\}$.

Proposition C.2.9. *Each Hausdorff normal topological space is Tychonoff.*

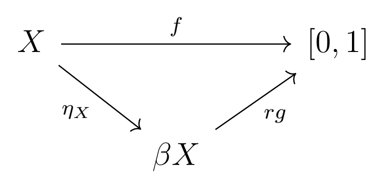
Proof. Suppose X is Hausdorff and normal. We need only be able to separate closed sets from points by continuous functions. However, since X is Hausdorff, points are closed and we may merely invoke Urysohn's lemma. QED

Proposition C.2.10. *If X is a Tychonoff space, then $\eta_X : X \to \beta X$ is an embedding into a dense subset of βX.*

Proof. We first prove that η_X is an injection. Suppose, $x, y \in X$ with $x \neq y$. Then since X is Tychonoff, there is a continuous $f : X \to [0,1]$ so that $f(x) \neq f(y)$. We factor this through βX to obtain $\widehat{f} : \beta X \to [0,1]$ with $f = \widehat{f}\eta_X$. We immediately see that $\eta_X(x) \neq \eta_X(y)$ as otherwise $f(x) = f(y)$. Thus, η_X is an injection.

We now prove that η_X is an embedding. Suppose there is some net α in X with $\eta_X(\alpha) \to \eta_X(x)$ for some $x \in X$. We aim to show $\alpha \to x$. Suppose not. We may identify and open neighborhood of X and a cofinal subset $I \subseteq \mathrm{dom}(\alpha)$ so that $\alpha(I) \cap U = \emptyset$. It follows that $x \notin \mathrm{cl}(\alpha(I))$. There is thus a continuous function $f : X \to [0,1]$ so that $f(x) = 1$ and $f(\alpha(I)) = \{0\}$. We factor this through βX to obtain $\widehat{f} : \beta X \to [0,1]$ with $f = \widehat{f}\eta_X$. Since $\eta_X(\alpha) \to \eta_X(x)$ and $I \subseteq \mathrm{dom}(\alpha)$ is cofinal, we have that $\eta_X(\alpha)|_I \to \eta_X(x)$. But then $\widehat{f}\eta_X(\alpha)|_I \to 1$ where this net takes constant value zero. This is impossible, so $\alpha \to x$. Therefore, η_X has continuous inverse out of its image.

We lastly prove that $\eta_X(X)$ is dense in βX. Suppose to contradiction that there is some $z \in \beta X$ with $z \notin \mathrm{cl}(\eta_X(X))$. Since βX is compact and Hausdorff, it is normal by Proposition C.1.7. We may thus find a continuous function $g : \beta X \to [0,1]$ so that $g(z) = 1$ and $g(\mathrm{cl}(\eta_X(X)) = 0$. For each $r \in (0,1]$, we have that rg is continuous. Let $f : X \to [0,1]$ be the constant map with value 0. Note that for any $r \in (0,1)$ that



commutes. This contradicts the fact that f factors through βX uniquely. Therefore, $\eta_X(X)$ is dense in βX. QED

Proposition C.2.11. *Any subspace of a compact Hausdorff space is Tychonoff.*

Proof. Let X be a compact Hausdorff space with subspace Y. Let $x \in Y$ and $S \subseteq Y \smallsetminus \{x\}$ closed. We then have some closed subset C of X so that $S = Y \cap C$. Necessarily, $x \notin C$. Since X is compact Hausdorff, X is normal and $\{x\}$ is closed. By Urysohn's lemma, we may find a continuous $f : X \to [0,1]$ so that $f(x) = 1$ and $f(C) = \{0\}$. The restriction $f|_Y$ witnesses that Y is Tychonoff. QED

C.3 A Tietze-like Extension Result

Lemma C.3.1. *If X is a Tychonoff space and $K, C \subseteq X$ are disjoint and compact and closed respectively, then there exists a continuous function $f : X \to [0,1]$ so that $f(K) = \{0\}$ and $f(C) = \{1\}$.*

Proof. Since X is Tychonoff, we may find for each $x \in K$ a continuous function $f_x : X \to [0,1]$ so that $f_x(x) = 0$ and $f_x(C) = \{1\}$. For each $x \in X$ define an open set $U_x = f_x^{-1}([0,1/2))$. The collection $\{U_x : x \in X\}$ is then an open cover of K. By compactness, we may find $x_1, ..., x_n \in X$ so that

$$K \subseteq \bigcup_{i=1}^{n} U_{x_i}.$$

Define continuous $h : X \to [0, 1]$ by

$$h(x) = \min\{f_{x_1}(x), ..., f_{x_n}(x)\}.$$

We have that $h(x) < \frac{1}{2}$ for each $x \in K$ since the $U_{x_1}, ..., U_{x_n}$ cover K. We then compose h with some continuous $r : [0, 1] \to [0, 1]$ so that $r([0, 1/2]) = \{0\}$ and $r(1) = 1$. Thus, $f = r \circ h$ is a continuous function $f : X \to [0, 1]$ so that $f(K) = \{0\}$ and $f(C) = \{1\}$. QED

Theorem C.3.2. *Suppose X is a Tychonoff space and $K \subseteq X$ is compact. Any continuous function $f : K \to \mathbb{R}$ may be extended to X.*

Proof. [12] This proof is incorrect. See annotation in hard copy or Math 5200 homework 5. We have that K is compact and f continuous. We therefore have that $|f| < M$ for some $M > 0$. We will proceed to recursively construct a sequence of continuous functions $g_n : X \to \mathbb{R}$ so that for each $n \in \mathbb{N}$,

(i) for each $x \in K$, we have $\left|f(x) - \sum_{k=0}^{n} g_k(x)\right| \leq \frac{2^n M}{3^n}$;

(ii) for each $x \in X \setminus K$ we have $|g_n(x)| < \frac{2^{n-1} M}{3^n}$

For the base case, let g_0 be the constant map at 0. Suppose that we have constructed $g_0, ..., g_n$. Define closed subsets of K by

$$A = (f - g_n)^{-1}\left(-\infty, -\frac{2^n M}{3^{n+1}}\right]$$

$$B = (f - g_n)^{-1}\left[\frac{2^n M}{3^{n+1}}, \infty\right).$$

We note that A and B are compact as they are closed subsets of a compact space. Using the preceding lemma, find

$$g_{n+1} : X \to \left[-\frac{2^n M}{3^{n+1}}, \frac{2^n M}{3^{n+1}}\right]$$

so that $g_{n+1}(x) = -\frac{2^n M}{3^{n+1}}$ for $x \in A$ and $g_{n+1}(x) = \frac{2^n M}{3^{n+1}}$ for $x \in B$. It is then simple to verify that (i) and (ii) hold.

[12] This proof method here is essentially the same as that used to prove the Tietze Extension Theorem in [Arm83]. Here, the condition that K is compact, instead of merely closed, accounts for the weakening of X from normal to Tychonoff.

Lastly, define $\widehat{f} : X \to \mathbb{R}$ by

$$\widehat{f}(x) = \sum_{k=0}^{\infty} g_n(x).$$

Condition (ii) tells us that this series converges uniformly by the Weierstrass M-test, and thus $\widehat{f}$ is continuous. Condition (i) guarantees that f and $\widehat{f}$ agree on K. QED

Corollary C.3.3. *Suppose X is a Tychonoff space with compact subsets K. Any continuous function $f : K \to \mathbb{C}$ may be extended to X.*

Proof. One need merely extend the real and imaginary parts of f independently. QED

Appendix D

Topological Vector Spaces

This appendix contains results on vector spaces and topological vector spaces supporting definitions and results in Chapter 4. This appendix mainly reproduces material from [NB10].

D.1 Vector Space Preliminaries

Definition D.1.1. A subset A of a vector space V is called *convex* when $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and $\lambda \in [0, 1]$.

Proposition D.1.2. *Let V and W are vector spaces.*

(a) If $\mathcal{A}$ is a collection of convex subsets of V, then $\bigcap \mathcal{A}$ is convex.

(b) If $\alpha \in \mathbb{K}$ and $A \subseteq V$ is convex, then αA is convex.

(c) If $A, B \subseteq V$ are convex, then so is $A + B$.

(d) If $f : V \to W$ is a linear map and $A \subseteq V$ is convex, then $f(A)$ is convex.

Proof. We consider each sub-proposition in turn.

(a) Suppose $\mathcal{A}$ is a collection of convex subsets of V. Let $v, w \in \bigcap \mathcal{A}$ and $\lambda \in [0, 1]$. Then for each $A \in \mathcal{A}$, we have $\lambda v + (1 - \lambda)w \in A$ so that $\lambda v + (1 - \lambda)w \in \bigcap \mathcal{A}$. Thus, $\bigcap \mathcal{A}$ is convex.

(b) Suppose $\alpha \in \mathbb{K}$ and $A \subseteq V$ is convex. Let $v, w \in A$ and $\lambda \in [0, 1]$. Then $\lambda v + (1 - \lambda)w \in A$ so that

$$\alpha(\lambda v + (1 - \lambda)w) = \lambda(\alpha v) + (1 - \lambda)(\alpha w) \in \alpha A$$

so that αA is convex.

(c) Let $v_1, w_1 \in A$ and $v_2, w_2 \in B$. Fix $\lambda \in [0, 1]$. We have that

$$\lambda(v_1 + v_2) + (1 - \lambda)(w_1 + w_2) = \lambda v_1 + (1 - \lambda)w_1 + \lambda v_2 + (1 - \lambda)w_2 \in A + B$$

from convexity of $A + B$.

(d) Suppose $f : V \to W$ is a linear map and $A \subseteq V$ is convex. Let $x, y \in f(A)$ and $\lambda \in [0, 1]$. There are then $v, w \in A$ so that $f(v) = x$ and $f(w) = y$. We then have

$$\begin{aligned} \lambda x + (1-\lambda)y &= \lambda f(v) + (1-\lambda)f(w) \\ &= f(\lambda v + (1-\lambda)w) \\ &\in f(A) \end{aligned}$$

since A is convex. We conclude that $f(A)$ is convex.

QED

Definition D.1.3. Given a vector space V and subset A of V, we define the *convex hull* of A to be the smallest convex set containing A and denote this by $\mathrm{co}(A)$. That is, if $\mathcal{C}$ denotes the collection of convex sets containing A,

$$\mathrm{co}(A) = \bigcap \mathcal{C}.$$

Equivalently, one may verify that

$$\mathrm{co}(A) = \left\{ \sum_{i=1}^{n} \lambda_i a_i \mid a_1, ..., a_n \in A \text{ and } \lambda_1, ..., \lambda_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

Proposition D.1.4. *Let V be a vector space.*

1. *If $A \subseteq B$, then $\mathrm{co}(A) \subseteq \mathrm{co}(B)$.*
2. *If $\mathcal{A}$ is a collection of subsets of V, then $\mathrm{co}\left(\bigcap \mathcal{A}\right) \subseteq \bigcap_{A \in \mathcal{A}} \mathrm{co}(A)$.*

Proof. Suppose $A \subseteq B \subseteq V$. Any convex subset of V containing B contains A. Thus, $A \subseteq \mathrm{co}(B)$. Since $\mathrm{co}(B)$ is convex, $\mathrm{co}(A) \subseteq \mathrm{co}(B)$.

Suppose $\mathcal{A}$ is a collection of subsets of V. By analogous reasoning to the above, we have $\bigcap \mathcal{A} \subseteq \bigcap_{A \in \mathcal{A}} \mathrm{co}(A)$ which is convex. The desired result follows. QED

Proposition D.1.5. *If $f : V \to W$ is a linear mapping of vector spaces and $A \subseteq V$, the $\mathrm{co}(f(A)) = f(\mathrm{co}(A))$.*

Proof. Since $\mathrm{co}(A)$ is convex, and linear maps preserve convexity, $f(\mathrm{co}(A))$ is convex and contains $f(A)$. Thus, $\mathrm{co}(f(A)) \subseteq f(\mathrm{co}(A))$. On the other hand, if $v \in f(\mathrm{co}(A))$, then there are $a_1, ..., a_n \in A$ and $\lambda_1, ..., \lambda_n \in \mathbb{R}_{\geq 0}$ so that $\sum_{i=1}^{n} \lambda_i = 1$ and $v = \sum_{i=1}^{n} \lambda_i f(a_i)$. Thus, $v \in \mathrm{co}(f(A))$. We conclude that $f(\mathrm{co}(A)) \subseteq \mathrm{co}(f(A))$ and $f(\mathrm{co}(A)) = \mathrm{co}(f(A))$ as desired. QED

Proposition D.1.6. *If A and B are subsets of a vector space V, then $\mathrm{co}(A) + \mathrm{co}(B) = \mathrm{co}(A + B)$.*

Proof. Since $A\,\mathrm{co}(A)$ and $\mathrm{co}(B)$ are convex and contain A and B, we have that $\mathrm{co}(A)+\mathrm{co}(B)$ is convex and $\mathrm{co}(A)+\mathrm{co}(B)\supseteq \mathrm{co}(A+B)$.

For the other containment, a generic element of $\mathrm{co}(A+B)$ is

$$\sum_{i=1}^{n} \lambda_i(a_i+b_i)$$

where each $a_i \in A$ and $b_i \in B$ and $\lambda_i \in [0,1]$ so that $\sum_{i=1}^{n} \lambda_i = 1$. Then,

$$\sum_{i=1}^{n} \lambda_i(a_i+b_i) = \sum_{i=1}^{n} \lambda_i a_i + \sum_{i=1}^{n} \lambda_i b_i \in \mathrm{co}(A)+\mathrm{co}(B)$$

as desired. QED

Proposition D.1.7. *If U, V, W are vector spaces, $B : U \times V \to W$ is a bilinear map, and $M \subseteq U$ and $N \subseteq W$, then*

$$\mathrm{co}(B(M\times N)) \supseteq B(\mathrm{co}(M)\times N).$$

Proof. Suppose $w \in B(\mathrm{co}(M)\times N)$. There is then some $v \in N$ and $u_1, ..., u_n \in M$ and $\lambda_1, ..., \lambda_n \in \mathbb{R}_{\geq 0}$ so that $\sum_{i=1}^{n} \lambda_i = 1$ and

$$\begin{aligned} w &= B\left(\sum_{i=1}^{n} \lambda_i u_i, v\right) \\ &= \sum_{i=1}^{n} \lambda_i B(u_i, v). \end{aligned}$$

Thus, $w \in \mathrm{co}(B(M\times N))$. QED

Definition D.1.8. If V is a vector space, a subset A of V is called *balanced* whenever, $\lambda A \subseteq A$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$.

Proposition D.1.9. *The convex hull of a balanced set is balanced.*

Proof. Suppose $A \subseteq V$ is balanced. Let $v \in \mathrm{co}(A)$. Then $v = \sum_{i=1}^{n} \lambda_i a_i$ for some $a_1, ..., a_n \in A$ and $\lambda_1, ..., \lambda_n \in \mathbb{R}_{\geq 0}$ so that $\sum_{i=1}^{n} \lambda_i = 1$. Let $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$. We then have that $\lambda a_i \in A$ for each $i = 1, ..., n$. Thus,

$$\lambda v = \sum_{i=1}^{n} \lambda_i(\lambda a_i) \in \mathrm{co}(A)$$

so that $\mathrm{co}(A)$ is balanced. QED

Definition D.1.10. If V is a vector space, a subset A of V is called *absolutely convex* whenever A is both convex and balanced.

Definition D.1.11. Given a vector space V and subset A of V, we define the *absolute convex hull* of A to be the smallest absolutely convex set containing A and denote this by $\Gamma(A)$. This may be explicitly given as

$$\Gamma(A) = \left\{ \sum_{i=1}^{n} \lambda_i a_i \mid a_1, ..., a_n \in A \text{ and } \lambda_1, ..., \lambda_n \in \mathbb{K} \text{ and } \sum_{i=1}^{n} |\lambda_i| \leq 1 \right\}.$$

Definition D.1.12. A subset A of a vector space V is called *absorbent* when for every $v \in V$ there is $R > 0$ so that whenever $\lambda \in \mathbb{K}$ with $|\lambda| \geq R$ we have $v \in \lambda A$. Equivalently, there is $S > 0$ so that what $|\lambda| \leq S$ we have $\lambda v \in A$.

Definition D.1.13. If V is a vector space, a functional $f : V \to \mathbb{R}$ is called *sublinear* when for all $v, w \in V$ and $\lambda \in \mathbb{R}_{\geq 0}$

1. $f(\lambda v) = \lambda f(v)$;
2. $f(v + w) \leq f(v) + f(w)$.

Definition D.1.14. If V is a vector space, a map $p : V \to \mathbb{R}$ is called a *seminorm* when for all $v, w \in V$ and $\lambda \in \mathbb{K}$

1. $p(v) \geq 0$;
2. $p(\lambda v) = |\lambda| p(v)$;
3. $p(v + w) \leq p(v) + p(w)$.

Notation D.1.15. If V is a vector space, we denote the dual of V by $\mathcal{L}(V)$.

Definition D.1.16. If V is a vector space and $A \subseteq V$, define $A^{\perp} = \{\varphi \in \mathcal{L}(V) : \varphi(A) = \{0\}\}$. Similarly, if $M \subseteq \mathcal{L}(V)$, define $M^{\perp} = \{v \in V : \varphi(v) = 0 \text{ for all } \varphi \in M\}$.

D.2 Topological Vector Spaces

Definition D.2.1. A *topological vector space* or TVS is a vector space V over $\mathbb{K}$ equipped with a topology so that scalar multiplication $\cdot : \mathbb{K} \times V \to V$ and vector addition $+ : V \times V \to V$ are continuous.

Proposition D.2.2. *If V is a TVS and U is a neighborhood of 0, then U is absorbent.*

Proof. Let $v \in V$. Consider $\mathbb{R}_{>0}$ with the reverse of the usual order. Let

$$D = \{(r, \lambda) : r \in \mathbb{R}_{>0} \text{ and } \lambda \in \mathbb{K} \text{ with } |\lambda| = r\}$$

ordered by the first coordinate. Let $\delta : D \to \mathbb{K}$ by $\delta(r, \lambda) = \lambda$. Certainly $\delta \to 0$ in $\mathbb{K}$. We thus have that $\delta v \to 0$ in V. Thus, $\delta v \in_{\text{ev}} U$. This means exactly that there is $r > 0$ so that whenever $|\lambda| \leq r$ we have $\lambda v \in U$. So U is absorbent. QED

Definition D.2.3. If V is a vector space and $A \subseteq V$, define $p_A : V \to [0, \infty]$, the *Minkowski functional induced by* A, by

$$p_A(v) = \inf\{r > 0 : v \in rA\}.$$

with the convention that $\inf \emptyset = \infty$.

Proposition D.2.4. *If V is a TVS and $A \subseteq V$ is an absolutely convex neighborhood of 0, then $p_A : V \to \mathbb{R}$ is a seminorm.*

Proof. Since A is a neighborhood of 0, we know that A is absorbent. Thus, p_A takes only real values.

Suppose $v \in V$ and $\lambda \in \mathbb{K}$. If $\lambda = 0$, then certainly $p_A(\lambda v) = 0$. Else, suppose $|\lambda| p_A(v) < t$. We then have that $|p_A(v)| < t/|\lambda|$. Thus, we have $v \in t/|\lambda| A$ and $|\lambda| v \in tA$. There is some scalar z with $|z| = 1$ so that $|\lambda| z = \lambda$. We then obtain $\lambda v \in tA$ since A is balanced. Thus, $p_A(\lambda v) \leq |\lambda| p_A(v)$. Next, suppose $t > 0$ is such that $\lambda v \in tA$. We then have that $v \in t/\lambda A = t/|\lambda| A$. We then have that $p_A(v) \leq t/|\lambda|$ so that $|\lambda| p_A(v) \leq t$. Therefore, $|\lambda| p_A(v) \leq p(\lambda v)$ as desired for p_A to be a seminorm.

Next, suppose $v, w \in V$. For every $\epsilon > 0$ we may find $\lambda, \mu > 0$ so that $\lambda \leq p_A(v) + \epsilon$ and $v \in \lambda A$ and $\mu \leq p_A(w) + \epsilon$ and $w \in \mu A$. We have that

$$\frac{\lambda}{\lambda + \mu} A + \frac{\mu}{\lambda + \mu} A \subseteq A$$

by convexity of A. Thus,

$$\lambda A + \mu A \subseteq (\lambda + \mu) A.$$

We obtain $w + v \in (\lambda + \mu) A$. Therefore

$$p_A(v + w) \leq p_A(v) + p_A(w) + 2\epsilon.$$

This holds for all ϵ, so $p_A(v + w) \leq p_A(v) + p_A(w)$.

We have now shown that p_A is a seminorm. QED

Remark D.2.5. If we relax the hypotheses to A being merely a convex neighborhood of 0, the only change is that p_A is a non-negative sublinear functional.

Lemma D.2.6. *If $f : V \to \mathbb{R}$ is a sublinear functional on a TVS V, then f is continuous if it is continuous at 0.*

Proof. Suppose f is continuous at 0. Let $v \in V$ and $\alpha : A \to V$ be a net in V converging to v. We have that $\alpha - v \to 0$ and since f is continuous at 0, we have $f(\alpha - v) \to 0$. For any $a \in A$,

$$\begin{aligned} f(\alpha_a) - f(v) &= f(\alpha_a - v + v) - f(v) \\ &\leq f(\alpha_a - v) \end{aligned}$$

and

$$\begin{aligned} f(v) - f(\alpha_a) &= f(v - \alpha_a + \alpha_a) - f(\alpha_a) \\ &\leq f(v - \alpha_a) \\ &= f(\alpha_a - v) \end{aligned}$$

so that $|f(v) - f(\alpha_a)| \leq f(\alpha_a - v)$. Since $f(\alpha_a - v) \to 0$, we have $f(\alpha) \to f(v)$. Therefore, f is continuous on V. QED

Proposition D.2.7. *If V is a TVS and $A \subseteq V$ is a convex neighborhood of 0, then $p_A : V \to \mathbb{R}$ is continuous.*

Proof. We have that p_A does not take ∞ as a value since A is absorbent. Since p_A is sublinear, to show p_A is continuous we need only show it is continuous at 0. Let α be a net in V with $\alpha \to 0$. Fix $\epsilon > 0$. We have that $\epsilon/2A$ is open. Therefore, $\alpha \in_{\text{ev}} \epsilon/2A$. If $v \in \epsilon/2A$, then we have that $p_A(v) \leq \epsilon/2$. Therefore, $p_A(\alpha) \in_{\text{ev}} (-\epsilon, \epsilon)$. Thus, $p_A(\alpha) \to 0$ and p_A is continuous at 0 and which means it is continuous everywhere. QED

D.3 Locally Convex Spaces

Definition D.3.1. A TVS is called *locally convex* when 0 has a neighborhood basis of convex sets.

Proposition D.3.2. *If V is a locally convex topological vector space then there is a basis of neighborhoods containing 0 consisting entirely of absolutely convex sets.*

Proof. Let U be a neighborhood of 0. Without loss of generality, we may assume U is convex. Let $m : \mathbb{K} \times V \to V$ be scalar multiplication. Since V is a TVS, we have that m is continuous and $m^{-1}(U)$ is open. We thus have some open $D \ni 0$ in $\mathbb{K}$ and $A \ni 0$ in V so that $D \times A \subseteq m^{-1}(U)$. Without loss of generality, we may take $D = \delta\mathbb{D}$ for some $\delta < 1$. We then note that

$$m(\delta\mathbb{D} \times A) = \bigcup_{0<|\epsilon|\leq|\delta|} \epsilon A$$

so that $m(\delta\mathbb{D}\times A)$ is open. Further, it is then clear that $m(\delta\mathbb{D}\times A)$ is balanced. Lastly, we have that $\text{co}(m(\delta\mathbb{D}\times A)) \subseteq U$ since U is convex. As the convex hull of a balanced set is balanced, we have that U contains a convex and balanced neighborhood of 0. It follows that there is a basis of neighborhoods containing 0 consisting entirely of absolutely convex sets. QED

Proposition D.3.3. *Suppose V is a locally convex TVS and $f : V \to \mathbb{K}$ is a linear functional. There is a continuous seminorm $p : V \to \mathbb{R}$ so that $|f| \leq p$ if and only if f is continuous.*

Proof. Suppose such continuous seminorm p exists. Let α be a net in V converging to 0. We have by continuity that $p(\alpha) \to 0$ in $\mathbb{R}$. It follows that $|f(\alpha)| \to 0$ in $\mathbb{R}$ and thus that $f(\alpha) \to 0$ in $\mathbb{K}$. Therefore, f is continuous at 0 and therefore continuous on all of V.

Conversely, suppose that f is continuous. Then $|f|$ is the desired continuous seminorm. QED

Lemma D.3.4. *If A and B are absolutely convex, then $\text{co}(A \cup B)$ is absolutely convex.*

Proof. This is certainly convex. Now, let $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$. If $v \in \text{co}(A \cup B)$, then $v = \gamma a + (1-\gamma)b$ for some $\gamma \in [0,1]$ and $a \in A$ and $b \in B$ since A and B are absolutely convex. We than have that $\lambda v = \gamma(\lambda a) + (1-\gamma)(\lambda b)$. Since A and B are absolutely convex, this is within $\text{co}(A \cup B)$. Thus, $\text{co}(A \cup B)$ is absolutely convex. QED

Proposition D.3.5. *If V is a locally convex TVS and M a subspace of V, then any continuous seminorm $p : M \to \mathbb{R}$ may be extended to V.*

Proof. Let $U = \{m \in M : p(m) < 1\}$ which is open in M and absolutely convex. Since M has the subspace topology, there is open $W'' \subseteq V$ so that $U = W'' \cap M$. We have that W'' contains 0, so there exists a neighborhood W' of 0 which is absolutely convex and contained in W''. We then have that $U \supseteq M \cap W'$. Let $W = \text{co}(U \cup W')$. We see that $U \subseteq M \cap W$ by its definition.

Now, suppose $v \in M \cap W$. Since U and W' are individually absolutely convex, we have some $\lambda \in [0,1]$ and $u \in U$ and $w' \in W$ so that $v = \lambda u + (1-\lambda)w'$. Thus, $(1-\lambda)w' = v - \lambda u \in M$. It then follows that $\lambda = 1$ or $w' \in U$. In both cases, $v \in U$. Therefore, $U = M \cap W$

Since U and W' are absolutely convex, we have that W is absolutely convex. This is absolutely convex neighborhood of 0 and is thus absorbent. We then consider the Minkowski functional $p_W : V \to \mathbb{R}$ by $p_W(v) = \inf\{r > 0 : v \in fA\}$ which is a continuous seminorm. We need only show that $p_W|_M = p$. We see that since $U = M \cap W$ that if $m \in M$,

$$p_W(m) = \inf\{r > 0 : m \in rU\} = \inf\{r > 0 : p(m) < r\} = p(m)$$

as desired. QED

D.4 The Hahn-Banach Theorem

Lemma D.4.1 (One Dimensional Extension)**.** *Suppose V is a vector space over $\mathbb{R}$ with subspace M. If there exists $v \in V \setminus M$ and a functional $f : M \to \mathbb{R}$ and a sublinear functional $p : V \to \mathbb{R}$ so that $f \leq p$ on M, then there exists an extension functional $\widehat{f}$ of f with $\widehat{f} : M \oplus \mathbb{R}v \to \mathbb{R}$ and $\widehat{f} \leq p$ on $M \oplus \mathbb{R}v$.*

Proof. For each $\delta \in \mathbb{R}$, define $\widehat{f_\delta} : M \oplus \mathbb{R}v \to \mathbb{R}$ by $\widehat{f_\delta}(m + rv) = f(m) + r\delta$ for all $m \in M$ and $r \in \mathbb{R}$. Each is clearly linear and an extension of f. We claim there is some $\delta \in \mathbb{R}$ making $\widehat{f_\delta} \leq p$.

Consider that for any $x, y \in M$, we have

$$\begin{aligned} f(x) + f(y) &= f(x + y) \\ &\leq p(x + y) \\ &= p(x - v + y + v) \\ &\leq p(x - v) + p(y + v) \end{aligned}$$

From this, we have that for any $x, y \in M$

$$f(x) - p(x - v) \leq p(y + v) - f(y).$$

We then have that for any $y \in M$,

$$\sup_{x \in M} f(x) - p(x - v) \leq p(y + v) - f(y)$$

and thus

$$\sup_{x \in M} f(x) - p(x - v) \leq \inf_{y \in M} p(y + v) - f(y)$$

We then choose δ so that

$$\sup_{x \in M} f(x) - p(x - v) \leq \delta \leq \inf_{y \in M} p(y + v) - f(y)$$

If $r > 0$ and $m \in M$, we then have that

$$r\delta \leq p(m + rv) - f(m)$$

so that $\widehat{f}(m+rv) \leq p(m+rv)$. If $r = 0$ then $\widehat{f}(m+rv) \leq p(m+rv)$ since $f(m) \leq p(m)$. If $r < 0$, then

$$-r\delta \geq f(m) - p(m + rv)$$

so that $\widehat{f}(m + rv) \leq p(m + rv)$. We conclude that $\widehat{f} \leq p$ as desired. QED

Theorem D.4.2 (Hahn-Banach, Real Case)**.** *Let V be a vector space over $\mathbb{R}$ with subspace M. If $f : M \to \mathbb{R}$ is some functional and $p : V \to \mathbb{R}$ is some sublinear functional so that $f \leq p$, then there exists a linear functional $\widehat{f} : V \to \mathbb{R}$ extending f such that $\widehat{f} \leq p$.*

Proof. Define

$$\mathcal{P} = \{\varphi : A \to \mathbb{R} \mid M \subseteq A \subseteq V \ \wedge \ A \text{ a subspace } \wedge \ \varphi \text{ linear } \wedge \ \varphi|_M = f \ \wedge \ \varphi \leq p\}$$

Given $\varphi, \psi \in \mathcal{P}$, say that $\varphi \leq \psi$ when the domain of φ is a subset of the domain of ψ and ψ restricts to φ. This makes $\mathcal{P}$ into a poset. Suppose $\mathcal{C} \subseteq \mathcal{P}$ is a chain. Let Ω be the union of the domains of elements of $\mathcal{C}$. This is a subspace of V since it is a union of nested subspaces. Define $\omega : \Omega \to \mathbb{R}$ by $\omega(x) = \varphi(x)$ where φ is any element of $\mathcal{C}$ whose domain contains x. This is well defined and linear since successive elements of $\mathcal{C}$ are linear and extend each other. By the same reasoning, $\omega \leq p$. By Zorn's lemma, $\mathcal{P}$ has a maximal element. Call it $\widehat{f} : A \to \mathbb{R}$. If $A \neq V$, then by the one dimensional extension lemma, $\widehat{f}$ is not maximal. So $A = V$ and $\widehat{f}$ is the desired extension of f. QED

Lemma D.4.3. *If V is a vector space over $\mathbb{C}$ and f a functional on V with real part f_1, then for all $v \in V$,*

$$f(v) = f_1(v) - i f_1(iv).$$

Proof. Let f_2 denote the imaginary part of f. For any $v \in V$, we have that $f(v) = f_1(v) + i f_2(v)$. Thus,

$$f(iv) = f_1(iv) + i f_2(iv)$$

and since f is $\mathbb{C}$-linear,

$$f(v) = f_2(iv) - i f_1(iv).$$

By the uniqueness of real and imaginary parts, $f_2(v) = -f_1(iv)$ for all $v \in V$. The result follows. QED

Corollary D.4.4 (Hahn-Banach, Complex Case)**.** *Let V be a vector space over $\mathbb{C}$ with subspace M. If $f : M \to \mathbb{C}$ is some functional and $p : V \to \mathbb{R}$ is some seminorm so that $|f| \leq p$, then there exists a linear functional $\widehat{f} : V \to \mathbb{C}$ extending f such that $|\widehat{f}| \leq p$.*

Proof. Let f_1 denote the real part of f. Since $|f| \leq p$, we have that $f_1 \leq p$. We view V as a real vector space and use the real version of Hahn-Banach to obtain an $\mathbb{R}$-linear extension $\widehat{f_1}$ of f_1 to V. We then set $\widehat{f}(x) = \widehat{f_1}(x) - i\widehat{f_1}(ix)$ for each $x \in V$. By the preceding lemma, this does restrict to f on M. One may then verify that $\widehat{f}$ is $\mathbb{C}$-linear. It is then left to verify that $|\widehat{f}| \leq p$. Choose $v \in V$. Let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ so that $\alpha f(v) \in \mathbb{R}$. We obtain

$$\begin{aligned} |\widehat{f}(v)| &= |\alpha \widehat{f}(v)| \\ &= |\widehat{f}(\alpha v)| \\ &= |\widehat{f_1}(\alpha v)| \\ &\leq p(\alpha v) \\ &= |\alpha| p(v) \\ &= p(v) \end{aligned}$$

as desired. QED

Corollary D.4.5. *If V is a locally convex TVS and M is a subspace of V, and $f : M \to \mathbb{K}$ is a continuous functional, then f has a continuous extension $\widehat{f}$ to V.*

Proof. Since f is continuous on a subspace of V, we have a continuous seminorm $p : V \to \mathbb{R}$ so that $|f| \leq p$. By Hahn-Banach, we extend f to a linear $\widehat{f} : V \to \mathbb{K}$ so that $|\widehat{f}| \leq p$. We thus have that $\widehat{f}$ is continuous. QED

Notation D.4.6. If V is a vector space and $p : V \to \mathbb{R}$ is a sublinear functional, let $V_p = \{v \in V : p(v) < 1\}$

Lemma D.4.7. *If V is a vector space with sublinear functional p*

1. *V_p is convex;*
2. *If $r > 0$ then $rV_p = \{v \in V : p(v) < r\}$;*
3. *If $w \in V$ then $w + V_p = \{v \in V : p(v - w) < 1\}$*

Proposition D.4.8. *If V is vector space over $\mathbb{R}$ and f is a non-trivial linear functional and $p : V \to \mathbb{R}$ is a non-negative sublinear functional, then*

1. *$f \leq p$ if and only if $f^{-1}(\{1\}) \cap V_p \neq 0$;*
2. *If V is a TVS, p is continuous, and $f \leq p$ then f is continuous.*

Proof. (1) is a straightforward double inclusion and (2) is not significantly different from the case wherein p is a seminorm; one need only check that $|f| \leq p$. QED

Proposition D.4.9. *If V is a TVS and p is a sublinear functional of V, then*

1. *$p(v) - p(w) \leq p(v - w)$ for all $v, w \in V$;*
2. *If p is non-negative, then p is continuous iff V_p is open;*
3. *If U is an open convex neighborhood of 0, then the Minkowski functional p_U is a non-negative, continuous, sublinear functional and $U = V_{p_U}$;*
4. *The open convex subsets of V are precisely those of the form $v + V_p$ for $v \in V$ and p a non-negative, continuous, sublinear functional.*

Proof. (1) This follows from

$$p(v) = p(v - w + w) \leq p(v - w) + p(w).$$

(2) This follows from the fact that V_p is open iff p is continuous at 0. Note that for any $\epsilon > 0$ we have $p^{-1}(-\epsilon, \epsilon) = \epsilon V_p$ since p is non-negative.

(3) Non-negativity is clear. That p_U is sublinear follows the fact that U is a convex balanced open neighborhood of 0. Suppose $v \in V_{p_U}$. We then have that $p_U(v) < 1$. It follows that for some $0 < \lambda < 1$ we have $v \in \lambda U \subseteq U$. Thus, $v \in U$. Now,

suppose that $v \in U$. Let $\delta : \mathbb{N} \to \mathbb{K}$ by $\delta_n = 1 + 1/n$. By continuity of scalar multiplication, $\delta v \to v$. We have that $\delta v \in_{\text{ev}} U$. Thus, there is $1 < t$ so that $tv \in U$. We then have that $v \in (1/t)U$ with $1/t < 1$. Therefore, $p_U(v) < 1$ and $v \in V_{p_U}$. We conclude that $U = V_{p_U}$. Since U is open, we have that p_U is continuous.

(4) Just translate convex open sets to 0 and apply part (3) and translate back. QED

Definition D.4.10. If V is a vector space, $H \subseteq V$ is called a *hyperplane* when there exists a non-trivial functional $f : V \to \mathbb{K}$ and scalar $\lambda \in \mathbb{K}$ and $H = \{v \in V : f(v) = \lambda\}$]. Such a hyperplane is denoted by $[f = \lambda]$.

Remark D.4.11. A hyperplane if closed if and only if its defining function is continuous.

Theorem D.4.12 (Hahn-Banach Geometric Form, Real Case)**.** *Let V be a real TVS with $M \subseteq V$ a subspace. If $U \subseteq V$ is open and convex and $M \cap U = \emptyset$, then there is a closed hyperplane $H \supseteq M$ so that $H \cap U = \emptyset$.*

Proof. Since G is open and convex, we find $z \in V$ and non-negative sublinear $p : V \to \mathbb{R}$ so that $U = z + V_p$. We notice that now we have $\left[z + V_p\right] \cap M = \emptyset$.

Define $f : M \oplus \mathbb{R}z \to \mathbb{R}$ by $f(m + az) = -a$ for some and $m \in M$ and $a \in \mathbb{R}$. We claim that $f \leq p$ on $M \oplus \mathbb{R}z$. Suppose otherwise, that is there is some $a \in \mathbb{R}$ and $m \in M$ so that $p(m + az) < -a$. Since p is non-negative, it must be that $-a$ is positive. We thus have that $p(m - z) < 1$. But then $m \in z + V_p$ which is impossible. We conclude that $f \leq p$. We may then extend f to V so that the extension $\widehat{f} \leq p$. It follows that $\widehat{f} : V \to \mathbb{R}$ is continuous.

Letting $H = \ker \widehat{f}$ we have $M \subseteq H$ as desired. Further, H is closed since $\widehat{f}$ is continuous. It is left to show that $H \cap U \neq 0$. Suppose $v \in H$. We then have

$$\begin{aligned} 1 &= \widehat{f}(-z) \\ &= \widehat{f}(x - z) \\ &\leq p(x - z) \end{aligned}$$

so that $x \notin z + V_p = U$. QED

Corollary D.4.13. *If V is a TVS and $U \subseteq V$ is convex, open and $M \subseteq V$ is a subspace, then there exists a continuous linear functional $g : V \to \mathbb{K}$ so that $g(M) = \{0\}$ and $0 \notin g(U)$.*

Proof. Viewing V as a real vector space, create $\widehat{f} : V \to \mathbb{R}$ as above. If $\mathbb{K} = \mathbb{R}$, set $g = \widehat{f}$ and we are done. Otherwise, we then define $g : V \to \mathbb{C}$ by $g(v) = \widehat{f}(v) - i\widehat{f}(iv)$. We claim that g is continuous. We know it is linear, so it is enough to show continuity at 0. Consider a net $\alpha \to 0$ in V. By continuity of $\widehat{f}$, addition, and scalar multiplication, $g(\alpha) \sim \widehat{f}(\alpha) - i\widehat{f}(i\alpha) \to 0$. Thus, g is the desired continuous functional. QED

Proposition D.4.14. *If V is a locally convex TVS and $A \subseteq V$ is a subspace, then*

$$(A^{\perp})^{\perp} = \overline{A}.$$

Proof. We first will prove that $\overline{A} \subseteq (A^{\perp})^{\perp}$. If $a \in \overline{A}$, there is a net $\alpha \to a$ entirely within A. Then, if $\varphi \in A^{\perp}$, we have that $\varphi(\alpha) = 0$ the constant 0 net. Thus, $\varphi(a) = 0$ since φ is continuous.

Next, we will show that $(A^{\perp})^{\perp} \subseteq \overline{A}$. Let $x \in (A^{\perp})^{\perp}$. Suppose to contradiction that $x \notin \overline{A}$. We may find an open convex neighborhood U of x with $U \cap \overline{A} = \emptyset$. We may then find a continuous linear functional $g : V \to \mathbb{K}$ so that $g(\overline{A}) = \{0\}$ and $0 \notin g(U)$. We are done since $g \in A^{\perp}$ and $g(x) \neq 0$ is a contradiction. QED

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